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ELECTROSTATIC FIELDS AND STORED ENERGY: A NEW PERSPECTIVE

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Abstract
The expression for the potential energy of point charges is developed along the line commonly found in text books. The stored energy in the field spreading over the infinite space is synonymous to the total potential energy of point charges that produces the field. This combined potential energy could be either a positive or a negative quantity. The energy stored in the part of the field which is distributed in any finite region is defined as the integration of the energy density over the volume of this region. The positive and negative value of the combined potential energy of all point charges suggests two different expressions for the density of energy stored in the field. A relation is developed between the stored energy in the electrostatic field and the potential energy of point charges located in the same region. The energy stored in the field inside a charge free region, can also be found by integrating the 'surface energy density' on the surface bounding the region.

Keywords Electrostatics fields, point charges, potential energy, volume and surface energy density

Introduction
We perform a thought experiment; bring point charges from infinity, one by one, and place them at distinct fixed points in free space. If a point charge Q is brought from infinity and placed at a point P the work done w is given as[1, 2, 3, 6]:

$$w = -Q.\int_{\infty}^{P} \mathbf{E} \bullet \mathbf{d}\boldsymbol{\ell} = Q.V \qquad (1)$$

Where, **E**indicates the distribution of electric field intensity andV indicates the scalar electric potential at the point P; the zero reference potential is taken to be at infinity. The fields, **E**andV, are due to all charges brought from infinity prior to the arrival of the charge Q. Further, our objective being the 'stored energy', any energy radiated[4] during the process of bringing these charges is ignored. In bringing an arbitrary n number of point charges the total work done W_n is attributed to as the potential energy of the group of n-number of point charges $\mathbb{E}_n$.

Potential energy of stationary point charges
Consider the infinite region of free space. We bring a point charge Q_1 from infinity and position it at a fixed point P_1 inside an arbitrary finite volume v in this region. The potential energy e being proportional to the product of the point charge and the potential at the point where the charge is placed, when the first charge Q_1 was brought and placed at the point P_1 the work done w_1 was zero and its potential energy e_1 remained at zero value, i.e.$e_1 = w_1 = 0$.

When the second charge Q_2 was brought and placed at another fixed point P_2 inside this volume v_o, the work done w_2 is the product of this charge and the potential $V_{1,2}$ at the point P_2, caused by the charge Q_1, already present at the point P_1; thus:

$$w_2 = Q_2.V_{1,2} \qquad (2)$$

Now, the potential at P_1, where the charge Q_1 is located, is no more zero as a potential $V_{2,1}$ has developed at this point due to the field of the newly arrived charge Q_2. Therefore, the new charge Q_2 and the existing charge Q_1simultaneously gained potential energye_2. This energy gain must be equal to the work done in bringing the second charge Q_2. Thus, we have:

$$e_2 = w_2 = k.Q_2.V_{1,2} + k.Q_1.V_{2,1} \qquad (3)$$

Therefore, from Equations (2) and (3):

$$Q_2.V_{1,2} = k.Q_1.V_{2,1} + k.Q_2.V_{1,2} \qquad (4)$$

Now, since:

$$Q_1.V_{2,1} = Q_2.V_{1,2} = \frac{Q_1.Q_2}{4\pi\varepsilon_o.R_{12}} \quad (5)$$

where R_{12} indicates the distance between the points P_1 and P_2.
Therefore, on solving Equation (4) for the constant of proportionality k, we get:

$$k = 1/2 \quad (6)$$

Equation (3) giving the work done w_2, for bringing the charge Q_2 is equal to the gain in the potential energy e_2 of the new charge as well as the one existing already, can thus be rewritten as follows:

$$e_2 = w_2 = \left[\underbrace{\frac{1}{2}.Q_2.V_{1,2}} + \overbrace{\frac{1}{2}.Q_1.V_{2,1}}\right] \quad (7)$$

Now, as the third point charge Q_3 is brought from infinity and placed at still another fixed point P_3 within the volume v_o; the work done w_3 is the product of this charge and the sum of the potentials $V_{1,3}$ and $V_{2,3}$ at the point P_3 caused by the charges Q_1 and Q_2, respectively. Therefore, we have:

$$w_3 = Q_3.\left(V_{1,3} + V_{2,3}\right) \quad (8)$$

It can be seen that the potential energy gained by this new charge Q_3 is:

$$\frac{w_3}{2} = \frac{1}{2}.Q_3.\left(V_{1,3} + V_{2,3}\right) \quad (9)$$

While, the other half of w_3 is stored as the combined increment in the potential energy of the previously brought charges, viz. Q_1 and Q_2, due to the potential field of the recently brought charge Q_3. This combined increment is as follows:

$$\frac{w_3}{2} = \left\{\frac{1}{2}.Q_1.V_{3,1} + \frac{1}{2}.Q_2.V_{3,2}\right\} \quad (10)$$

Thus, from Equations (9) and (10) the work done w_3, for bringing the charge Q_3 is equal to the gain in the potential energy e_3 of the new charge Q_3 as well as those existing already, i.e.Q_1 and Q_2:

$$e_3 = w_3 = \left[\underbrace{\frac{1}{2}.Q_3.\left(V_{1,3} + V_{2,3}\right)} + \overbrace{\frac{1}{2}.Q_1.V_{3,1} + \frac{1}{2}.Q_2.V_{3,2}}\right] \quad (11)$$

Now, the total work done W_3 to bring and position the three point charges and thus the total potential energy $\mathbb{E}_3$ of these charges is:

$$W_3 = w_1 + w_2 + w_3 = e_1 + e_2 + e_3 \stackrel{\text{def}}{=} \mathbb{E}_3 \quad (12)$$

or, $W_3 = \mathbb{E}_3 = \left[\frac{1}{2}.Q_1.0\right] + \left[\underbrace{\frac{1}{2}.Q_2.V_{1,2}} + \overbrace{\frac{1}{2}.Q_1.V_{2,1}}\right]$

$$+ \left[\underbrace{\frac{1}{2}.Q_3.\left(V_{1,3} + V_{2,3}\right)} + \overbrace{\frac{1}{2}.Q_1.V_{3,1} + \frac{1}{2}.Q_2.V_{3,2}}\right] \quad (13)$$

On rearranging terms on its R.H.S. we get:

$$W_3 = \mathbb{E}_3 = \frac{1}{2}.\left[Q_1.\left(V_{2,1} + V_{3,1}\right) + Q_2.\left(V_{1,2} + V_{3,2}\right) + Q_3.\left(V_{1,3} + V_{2,3}\right)\right]$$

or, $W_3 = \mathbb{E}_3 \stackrel{\text{def}}{=} \frac{1}{2}.[V_1.Q_1 + V_2.Q_2 + V_3.Q_3]$ (14)

where, the symbols V_1, V_2 and V_3, indicate the potentials at points P_1, P_2 and P_3 respectively, due to all charges except those located at respective points.
Now, Equation (14) can also be written as:

$$W_3 = \mathbb{E}_3 = \frac{1}{2}.\sum_{m=1}^{3} V_m^{(3)}.Q_m \quad (15)$$

where, the symbol Q_m indicates the m^{th} point charge brought from infinity and placed at the distinct fixed point P_m in the region of an arbitrary finite volume v_o, while $V_m^{(3)}$ indicates the potential at the point P_m due to all charges except the point charge Q_m.
For n number of point charges Equation (15) is modified as follows:

$$W_n = \boxed{\mathbb{E}_n = \frac{1}{2}.\sum_{m=1}^{n} V_m^{(n)}.Q_m} \quad (16)$$

the symbol $V_m^{(n)}$ indicates the potential at the point P_m due to all n- charges except the charge Q_m, located at the point P_m.
Equation (16) gives the total work done W_n in terms of the point charges and their potential fields. The same is also the total potential energy $\mathbb{E}_n$ attained by all point charges brought from infinity. **Since a charge and its potential could be either a positive or a negative quantity, the combined potential energy $\mathbb{E}_n$ could also be either a positive or a negative quantity**.

Since V_1 is zero if n is one, Equation (16) shows that the potential energy of an isolated single point charge$\mathbb{E}_1$ is zero.

Energy expression in terms of field quantities

Consider the delta-Dirac function defined as follows:

$$\begin{cases} \delta(x-x_0) \stackrel{\text{def}}{\equiv} 0 \quad \text{for } x \neq x_0 \\ \text{and } \int_{-\infty}^{+\infty} \delta(x-x_0).dx \stackrel{\text{def}}{\equiv} 1 \end{cases} \quad (17a)$$

Thus, for an arbitrary function $F(x)$, which is continuous at $x = x_0$:

$$\int_{x_1}^{x_2} F(x).\delta(x-x_0).dx = \begin{cases} 0, & \text{for } x_1 < x_2 < x_0 \\ F(x_0), & \text{for } x_1 < x_0 < x_2 \\ 0, & \text{for } x_0 < x_1 < x_2 \end{cases} \quad (17b)$$

Using delta-Dirac functions point charges can be expressed as a distribution of volume charge density ρ; as given below:

$$\rho = \sum_{m=1}^{n} Q_m .\delta(x-x_m).\delta(y-y_m).\delta(z-z_m) \quad (17c)$$

where, x_m, y_m and z_m indicates the coordinates of the point P_m.

Thus Equation (16) can be transformed as follows:

$$W_n = \mathbb{E}_n = \boxed{\frac{1}{2}.\sum_{m=1}^{n} V_m.Q_m \rightleftarrows \frac{1}{2}.\iiint_{v\to\infty}(V.\rho).dv} \quad (18)$$

This equation is identically satisfied if we substitute the expression for the charge density ρ from Equation (17) on the R.H.S. of Equation (18) and then the integration is performed over infinite volume, as shown below:

$$\frac{1}{2}.\iiint_{v\to\infty}(V.\rho).dv = \frac{1}{2}.\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty}\int_{z=-\infty}^{\infty}\sum_{m=1}^{n} f_m .dx.dy.dz \quad (19)$$

or, $\frac{1}{2}.\iiint_{v\to\infty}(V.\rho).dv = \frac{1}{2}.\sum_{m=1}^{n}\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty}\int_{z=-\infty}^{\infty} f_m.dx.dy.dz$

or, $\frac{1}{2}.\iiint_{v\to\infty}(V.\rho).dv = \frac{1}{2}.\sum_{m=1}^{n} V(x_m,y_m,z_m).Q_m \stackrel{\text{def}}{\equiv} \frac{1}{2}.\sum_{m=1}^{n} V_m.Q_m$

where, $f_m \stackrel{\text{def}}{\equiv} V(x,y,z).Q_m.\delta(x-x_m).\delta(y-y_m).\delta(z-z_m)$ (20)

The integration over infinite volume ensures that all n point charges brought from infinity and placed at distinct stationary points are included in the integration. On the other hand, if the integration is performed over an arbitrary finite volume v_ℓ, containing only ℓ number of point charges while some charges, viz. $(n-\ell)$ are left outside this volume; the resulting volume integration will give the total potential energy $\mathbb{E}_\ell$ of only those ℓnumber of pointcharges located within the volume v_ℓ. With point charges labeled afresh, Equation (18) is modified for the arbitrary volume v_ℓ as follows:

$$\boxed{\mathbb{E}_\ell = \frac{1}{2}.\sum_{k=1}^{\ell} V_k^{(n)}.Q_k = \frac{1}{2}.\iiint_{v_\ell}(V.\rho).dv} \quad (21)$$

where, $0 \le \ell \le n$. (21a)

and the symbol $V_k^{(n)}$ indicates the potential at the point P_k in the volume v_ℓ due to all n- charges, internal as well as external, except the charge Q_k, located at the point P_k in the volume v_ℓ.

Note that if $n > 1$, $\mathbb{E}_\ell|_{\ell=1} \neq 0$. Further, **if no point charge is located inside the finite arbitrary volume v_ℓ, i.e. for charge free regions the volume integration in Equation (21) being zero:**

$$\mathbb{E}_\ell|_{\ell=0} = \frac{1}{2}.\iiint_{v_\ell}(V.\rho).dv\Big|_{\ell=0} = 0 . \quad (21b)$$

Using the Maxwell's equation[5]: $\rho = \underbrace{\nabla \bullet \mathbf{D}}$,Equation (21) can be rewritten as:

$$\mathbb{E}_\ell = \frac{1}{2}.\sum_{k=1}^{\ell} V_k^{(n)}.Q_k = \frac{1}{2}.\iiint_{v_\ell}\left(V.\underbrace{\nabla \bullet \mathbf{D}}\right).dv \quad (22)$$

Next, consider the following identity:

$$(V.\nabla \bullet \mathbf{D}) \equiv (-\nabla V) \bullet \mathbf{D} + \nabla \bullet (V\mathbf{D}) \quad (23)$$

Since, for electrostatic fields:

$$-\nabla V = \mathbf{E}, \quad (23a)$$

we get, in view of Equation (23):

$$(V.\rho) = (V.\nabla \bullet \mathbf{D}) = \mathbf{E} \bullet \mathbf{D} + \nabla \bullet (V\mathbf{D}) \quad (24)$$

Therefore Equation (22) can be rewritten as:

$$\mathbb{E}_{\ell} = \frac{1}{2}.\sum_{k=1}^{\ell} V_k^{(n)}.Q_k = \iiint_{v_\ell}\left(\frac{1}{2}.\mathbf{E}\bullet\mathbf{D}\right).dv + \iiint_{v_\ell}\boldsymbol{\nabla}\bullet\left(\frac{1}{2}.V\mathbf{D}\right).dv \quad (25)$$

Equation (25) gives the potential energy of point charges exclusively in terms of field quantities. **This prompts one toconclude** that "the potential energy of point charges may also be considered as stored in the electrostatic fields".

The combined potential energy of all the point charges brought from infinity, in view of Equations (18) and (25) is given as follows:

$$\boxed{\mathbb{E}_n = \iiint_{v\to\infty}\left(\frac{1}{2}.\mathbf{E}\bullet\mathbf{D}\right).dv + \iiint_{v\to\infty}\boldsymbol{\nabla}\bullet\left(\frac{1}{2}.V\mathbf{D}\right).dv} \quad (26)$$

Using Gauss theorem, the second term on the R.H.S. of this equation is transformed into a surface integration, i.e.:

$$\iiint_{v\to\infty}\boldsymbol{\nabla}\bullet\left(\frac{1}{2}.V\mathbf{D}\right).dv \mathrel{\overrightarrow{\overrightarrow{=}}} \oiint_{s\to\infty}\left(\frac{1}{2}.V\mathbf{D}\right)\bullet\mathbf{ds} \stackrel{\text{def}}{=} \oiint_{s\to\infty}\boldsymbol{\mathcal{E}_s}\bullet\mathbf{ds} \quad (27)$$

Where the symbol s stands for the surface of the volume v and the surface density of the stored energy is defined as:

$$\boldsymbol{\mathcal{E}_s} \stackrel{\text{def}}{=} \left(\frac{1}{2}.V\mathbf{D}\right) \quad (27a)$$

As the volume v increases its surface s also increases; the average distance r of this surface from point charges also increases. Point charges being the source for the electric field; on the surface s, V varies as $1/r$ and **D**as $1/r^2$. Hence ($V\mathbf{D}$) in the expression of $\boldsymbol{\mathcal{E}_s}$ as given in the Equation (29c), must vary as $1/r^3$. Since s varies as r^2, with the increase in r the integrand decreases at a faster rate than the rate of increase in the surface area. Therefore, as the surface tends to infinity, the integration over infinite surface must tend to zero[1, 2, 6], i.e.:

$$\boxed{\oiint_{s\to\infty}\boldsymbol{\mathcal{E}_s}\bullet\mathbf{ds} \mathrel{\overrightarrow{\overrightarrow{=}}} 0} \quad (27b)$$

Therefore, Equation (26) reduces to:

$$\boxed{\mathbb{E}_n = \iiint_{v\to\infty}\left(\frac{1}{2}.\mathbf{E}\bullet\mathbf{D}\right).dv} \quad (28)$$

In view of above equation, the classical expression for the density of the energy stored in the electrostatic fields $\mathcal{E}$, is **defined**[6] as indicated below:

$$\mathcal{E} \stackrel{\text{def}}{=} \frac{1}{2}.\mathbf{E}\bullet\mathbf{D} \quad (28a)$$

Thus, $\boxed{\mathbb{E}_n = \iiint_{v\to\infty}\mathcal{E}.dv}$ (28b)

Conclusion

The statement that "the potential energy of point charges may also be considered as stored in the electrostatic fields" is speculative in nature rather than a fact. Consider the following observations:

(i) If energy is stored in the electrostatic field spreading over infinite space, we must be able to estimate the energy stored in any arbitrary finite volume for a given field distribution therein. The expression for the energy density defined by Equations (28a) and (28b) is valid only for the region of infinite volume.

(ii) When the first charge Q_1 was brought from infinity no energy was stored, even though electrostatic field is established in the infinite space surrounding the point charge.

(iii) In view of Equation (16), it has been argued that since a charge and its potential could be either a positive or a negative quantity, the combined potential energy $\mathbb{E}_n$ could also be either a positive or a negative quantity. The energy density defined by Equations (28a) is nowhere negative, therefore Equations (28b) gives erroneous result if the combined potential energy $\mathbb{E}_n$ is negative.

The dilemma presented above leads to the conclusion that no energy is stored in electrostatic field even though the potential energy of point charges, if it is positive, is expressible in terms of field quantities. If this is true, what about the time varying electromagnetic fields?

In general, if a point charge is shifted to a new position, it gains a potential energy which is equal to the combined gain in the potential energy of all remaining point charges. Therefore, energy needed to shift a point charge is double the energy gained by the shifted point charge. This energy is added to the energy already stored in the electrostatic field. Therefore, if some energy is to be extracted from or implanted into the electrostatic field of a set of point charges, the potential energy of one or more point charges may be changed by suitable displacements. The agency shifting the point charges needs to supply an extra energy which is radiated during the shift.

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