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Stochastic bi-level programming for Stackelberg game: a comparative study

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Abstract: This paper is designed based on a computational algorithm for stochastic bi-level programming in Stackelberg game using fuzzy optimization technique. It includes all the parameters are fuzzy numbers except the right side of the constraints which follow a normal distribution. Mellin's transformation is considered to transform the fuzzy numbers into crisp value. Using the stochastic programming technique, all the probabilistic constraints are converted into deterministic constraints. In addition, a comparative study of linear and non-linear membership functions for the degree of acceptance corresponding to LINGO 15.0 iterative scheme and genetic algorithm (GA) to see its contact on optimization. In order to show the efficient of the algorithm, a numerical example is presented.
Finally, conclusion about the findings and outlook are described.

Keywords: *Bi-level programming, Fuzzy number, GA, Stackelberg game, Stochastic programming.*

1. Introduction

Bi-level programming problem (BLPP) was first initiated by H. von Stackelberg, is a nested optimization problem including a leader and a follower problem and it has acknowledged significant attention of many researchers in the past few decades. Each decision maker in BLPP optimizes his/her own objective function without considering the objective function of the other, but the assessment of each effects the objective value of the other party. In Stackelberg game, the leader has the ability to implement his/her decision on the followers. Many researchers have been developed on Stackelberg game (cf. [1], [12], [14]). In real-life problems, it is extremely tricky to know all the information regarding the input parameters of the mathematical programming model because applicable data are scarce or in-existent, intricate to obtain or to estimate, the system is subject to changes, and so forth, that is, input parameters are uncertain. In uncertainty theory, there were two classes of uncertainties: fuzziness and randomness. Based on these classes, fuzzy programming has been urbanized by the parameters as fuzzy sets whereas stochastic programming has been measured by the parameters as random variables. Using the Mellin's transformation based on the proportional probability density function, all fuzzy numbers (triangular or trapezoidal) are converted to a crisp value. The right side of the constraints follows a normal distribution. After that, using a stochastic programming technique, the probabilistic constraints are transformed into the deterministic form and the corresponding crisp problems solved by LINGO 15.0 iterative scheme and GA. This paper includes the following:

- Mellin's transformation is considered for getting the crisp value from the fuzzy numbers.
- The right side parameters of the constraints are considered as a normal distribution.
- A computational algorithm designed to solve a Stackelberg game.

The inspiration of the present study is to give computational algorithm for solving a stochastic bi-level programming in Stackelberg game involving a normal distribution by fuzzy programming technique. It also includes the impacts of various type of membership function in such optimization process and thus have made comparative study of linear membership function with that of non-linear function for membership. The research works of several authors related to this area are shown in Table 1.

Table 1: Research works of several authors related to this area.

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Author(s)	Multi-choice	Fuzzy random	Fuzzy number	Real number
Anandalingam and White [1]				yes
Anandalingam and Apprey [2]	yes			
Hamidi and Mishmast [5]			yes	
Maity and Roy [9]	yes		yes	
Roy and Maiti [14]	yes	yes		
Sakawa and Matsui [20]		yes		
Singh and Chakraborty [21]		yes		
Yeh et al. [22]		yes		
Youness et al. [23]			yes	
Zheng et al. [25]				yes
Our paper		yes	yes	

The rest of the paper is organized as follows: Section 2 introduces the preliminaries basic concept of fuzzy number. In Section 3, the formulation of a stochastic bi-level programming in Stackelberg game involving a normal distribution is presented. The computation algorithm of fuzzy optimization technique is discussed in Section 4 and the usefulness of this algorithm is illustrated by a numerical example in Section 5. In Section 6, results and discussion about example is presented. The conclusion and an outlook of future study are described in Section 7.

2. Preliminaries

Here, we recall some useful definitions of fuzzy number and Mellin's transformation, which will be required for our subsequent developments.

Definition 2.1 [23] A fuzzy set $\tilde{A}$, defined on the universal set X is the family $\tilde{A}=\{(x,\mu_{\tilde{A}}(x)):x\in X\}$ where, $\mu_{\tilde{A}}:X\to[0,1]$ is the membership function such that $\mu_{\tilde{A}}(x)=0$ if x does not belong to $\tilde{A}$, $\mu_{\tilde{A}}(x)=1$ if x strictly belongs to $\tilde{A}$.

Definition 2.2 [5] A fuzzy set $\tilde{A}$, defined on the universal set of real numbers $\square$, is said to be a fuzzy number if its membership function has the following characteristics:

(a) $\mu_{\tilde{A}}(x):\square\to[0,1]$ is continuous,

(b) $\mu_{\tilde{A}}(x)=0$ for all $x\in(-\infty,a]\cup[d,\infty)$,

(c) $\mu_{\tilde{A}}(x)$ is strictly increasing on $[a,b]$ and strictly decreasing on $[c,d]$, $(a<b<c<d)$,

(d) $\mu_{\tilde{A}}(x)=\rho$, for all $x\in[b,c]$, where $0<\rho\le 1$.

Definition 2.3 [4] Let $\tilde{A}=(a_1,a_2,a_3)$ be a fuzzy number. Then $\tilde{A}$ is called a triangular fuzzy number with a piecewise linear membership function $\mu_{\tilde{A}}(x)$ is given by

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$$\mu_{\tilde{A}}(x)=\begin{cases}\left(\dfrac{x-a_1}{a_2-a_1}\right), & if\ a_1\le x\le a_2,\\ \left(\dfrac{a_3-x}{a_3-a_2}\right), & if\ a_2\le x\le a_3,\\ 0, & otherwise.\end{cases} \qquad (1)$$

Definition 2.4 [4] Let $\tilde{B}=(b_1,b_2,b_3,b_4)$ be a fuzzy number. Then $\tilde{B}$ is called a trapezoidal fuzzy number with a piecewise linear membership function $\mu_{\tilde{B}}(x)$ is given by

$$\mu_{\tilde{B}}(x)=\begin{cases}\left(\dfrac{x-b_1}{b_2-b_1}\right), & if\ b_1\le x\le b_2,\\ 1, & if\ b_2\le x\le b_3,\\ \left(\dfrac{b_4-x}{b_4-b_3}\right), & if\ b_3\le x\le b_4,\\ 0, & otherwise.\end{cases} \qquad (2)$$

Definition 2.5 [3] The Mellin's transform $M_X(t)$ of a probability density function $g(x)$, where x is positive, is defined as

$$M_X(t)=\int_0^\infty x^{t-1}g(x)\,dx$$

where the integral exists.

3. Model formulation

This paper presents a stochastic BLPP for Stackelberg game in which cost coefficients of the both levels objective functions are fuzzy numbers may be triangular or trapezoidal. In addition, the coefficients of the constraints in BLPP are considered as crisp. But the parameters of the constraints are imprecise in real-life situation. As a result, we consider the parameter of the constraints is two different natures. The left side of the constraints is a fuzzy number where as the right side follow a normal distribution. To prepare the model we impose the following assumption:

- $\tilde{c}_{1j}$: fuzzy number for the upper-level objective function, $j=1,2,....,n.$
- $\tilde{c}_{2j}$: fuzzy number for the lower-level objective function, $j=1,2,....,n.$

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- $\tilde{a}_{ij}$: a $m \times n$ matrix with fuzzy numbers.
- $\tilde{b}_i$: random variable follows a normal distribution.

Subsequently the mathematical model on a stochastic BLPP for Stackelberg game relating a normal distribution is as follows:

Model 1

$$\begin{aligned} &\text{max}imize \quad Z_1(x) = \sum_{j=1}^{n} \tilde{c}_{1j} x_j \\ &\text{max}imize \quad Z_2(x) = \sum_{j=1}^{n} \tilde{c}_{2j} x_j \\ &subject\ to \quad \Pr\left(\sum_{j=1}^{n} \tilde{a}_{ij} x_j \le \bar{b}_i \right) \ge \gamma_i, \quad i = 1, 2, ..., m \\ &\qquad x_j \ge 0, \forall\ j\ and\ 0 < \gamma_i < 1, \forall\ i. \end{aligned} \tag{3}$$

3.1 Crisp form of a fuzzy number

In this section, we discuss a defuzzification technique of a trapezoidal or triangular fuzzy numbers.

3.1.1 Defuzzification of a trapezoidal fuzzy number

Let $\tilde{A} = (A_1, A_2, A_3, A_4)$ be the trapezoidal fuzzy number, the proportional probability density function corresponding to $\tilde{A}$ is defined as follows: $f_{\tilde{A}}(x) = c\mu_{\tilde{A}}(x)$, where $\mu_{\tilde{A}}(x)$ is given by equation (2) and c is given by relation $\int_{-\infty}^{\infty} f_{\tilde{A}}(x)dx = 1$ (4)

From the equation (4), we obtain $c = \dfrac{2}{A_4 + A_3 - A_2 - A_1}$

Therefore, the proportional probability density function corresponding to $\tilde{A}$ is given by

$$f_{\tilde{A}}(x) = \begin{cases} \dfrac{2(x - A_1)}{(A_2 - A_1)(A_4 + A_3 - A_2 - A_1)}, & if\ A_1 \le x \le A_2 \\ \dfrac{2}{(A_4 + A_3 - A_2 - A_1)}, & if\ A_2 \le x \le A_3 \\ \dfrac{2(A_4 - x)}{(A_4 - A_3)(A_4 + A_3 - A_2 - A_1)}, & if\ A_3 \le x \le A_4 \\ 0, & otherwise \end{cases}$$

According to Definition 2.5, we obtain

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$$M_{\tilde{A}}(t)=\int_{0}^{\infty} x^{t-1} f_{\tilde{A}}(x)dx$$
$$=\int_{A_1}^{A_2} x^{t-1}\frac{2(x-A_1)}{(A_2-A_1)(A_4+A_3-A_2-A_1)}dx+\int_{A_2}^{A_3} x^{t-1}\frac{2}{(A_4+A_3-A_2-A_1)}dx$$
$$+\int_{A_3}^{A_4} x^{t-1}\frac{2(A_4-x)}{(A_4-A_3)(A_4+A_3-A_2-A_1)}dx$$

On integration, we obtain

$$M_{\tilde{A}}(t)=\frac{2}{(A_4+A_3-A_2-A_1)t(t+1)}\left[\frac{(A_4^{t+1}-A_3^{t+1})}{A_4-A_3}-\frac{(A_2^{t+1}-A_1^{t+1})}{A_2-A_1}\right].$$

Then, the mean $\left(E\left[\tilde{A}\right]\right)$ and variance $\left(\sigma_{\tilde{A}}^2\right)$ corresponding to $\tilde{A}$ are obtained as follows:

$$E\left[\tilde{A}\right]=M_{\tilde{A}}(2)$$
$$=\frac{1}{3}\left[(A_1+A_2+A_3+A_4)+\frac{(A_1A_2-A_3A_4)}{(A_4+A_3-A_2-A_1)}\right]$$
$$\sigma_{\tilde{A}}^2=M_{\tilde{A}}(3)-\left[M_{\tilde{A}}(2)\right]^2$$
$$=\frac{1}{6}\left[(A_1^2+A_2^2+A_3^2+A_4^2)+\frac{(A_1+A_2)(A_3^2+A_4^2)-(A_3+A_4)(A_1^2+A_2^2)}{(A_4+A_3-A_2-A_1)}\right]$$

Then the crisp value of $\tilde{A}$ is given by

$$Crisp(\tilde{A})=A'$$
$$=\frac{1}{3}\left[(A_1+A_2+A_3+A_4)+\frac{(A_1A_2-A_3A_4)}{(A_4+A_3-A_2-A_1)}\right] \quad (5)$$

3.1.2 Defuzzification of a triangular fuzzy number

Let $\tilde{B}=(B_1,B_2,B_3)$ be the triangular fuzzy number. In that case the crisp value is obtained by finding the expected value using the probability density function corresponding to the membership function.

Therefore, the proportional probability density function corresponding to $\tilde{B}$ is defined as follows: $f_{\tilde{B}}(x)=d\mu_{\tilde{B}}(x)$, where $\mu_{\tilde{B}}(x)$ is given by equation (1) .

To find d such that $\int_{-\infty}^{\infty} f_{\tilde{B}}(x)dx=1$ and we obtained $d=\frac{2}{B_3-B_1}$.

Then the proportional probability density function corresponding to $\tilde{B}$ is given by

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$$f_{\tilde{B}}(x)=\begin{cases}\dfrac{2(x-B_1)}{(B_2-B_1)(B_3-B_1)}, & if\ B_1\le x\le B_2\\ \dfrac{2(B_3-x)}{(B_3-B_2)(B_3-B_1)}, & if\ B_2\le x\le B_3\\ 0, & otherwise\end{cases}$$

According to Definition 2.5, we obtain

$$M_{\tilde{B}}(t)=\int_0^{\infty}x^{t-1}f_{\tilde{B}}(x)dx$$

$$=\int_{B_1}^{B_2}x^{t-1}\frac{2(x-B_1)}{(B_2-B_1)(B_3-B_1)}dx+\int_{B_2}^{B_3}x^{t-1}\frac{2(B_3-x)}{(B_3-B_2)(B_3-B_1)}dx$$

On integration, we obtain

$$M_{\tilde{B}}(t)=\frac{2}{(B_3-B_1)t(t+1)}\left[\frac{B_3(B_3^t-B_2^t)}{B_3-B_2}-\frac{B_1(B_2^t-B_1^t)}{B_2-B_1}\right].$$

Therefore, the mean $\left(E\left[\tilde{B}\right]\right)$ and variance $\left(\sigma_{\tilde{B}}^2\right)$ corresponding to $\tilde{B}$ are obtained as follows:

$$E\left[\tilde{B}\right]=M_{\tilde{B}}(2)$$

$$=\frac{B_1+B_2+B_3}{3}$$

$$\sigma_{\tilde{B}}^2=M_{\tilde{B}}(3)-\left[M_{\tilde{B}}(2)\right]^2$$

$$=\frac{B_1^2+B_2^2+B_3^2-B_1B_2-B_2B_3-B_3B_1}{18}$$

Then the crisp value of $\tilde{B}$ is given by

$$Crisp\left(\tilde{B}\right)=B'$$

$$=\frac{B_1+B_2+B_3}{3} \qquad (6)$$

Using the equation (5) (or equation (6)), Model 1 is transformed into the following model with the probabilistic constraint as

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Model 2

$$\begin{aligned} &\text{max}imize \quad Z_1(x) = \sum_{j=1}^{n} c'_{1j} x_j \\ &\text{max}imize \quad Z_2(x) = \sum_{j=1}^{n} c'_{2j} x_j \\ &subject\ to \quad \Pr\left(\sum_{j=1}^{n} a'_{ij} x_j \le \bar{b}_i \right) \ge \gamma_i, \quad i = 1, 2, ..., m \\ &\qquad x_j \ge 0, \forall\ j\ and\ 0 < \gamma_i < 1, \forall\ i. \end{aligned}$$

3.2 Deterministic form of probabilistic constraint

In this paper, $\bar{b}_i$, $\forall i$ are independent random variables which follows a normal distribution with mean and variance are b'_i and $Var\left(\bar{b}_i\right)$ respectively.

Then the constraint of Model 2 can be rewritten as follows:

$$\Pr\left(\sum_{j=1}^{n} a'_{ij} x_j \le \bar{b}_i \right) = \Pr\left(\frac{\sum_{j=1}^{n} a'_{ij} x_j - b'_i}{\sqrt{Var\left(\bar{b}_i\right)}} \le \frac{\bar{b}_i - b'_i}{\sqrt{Var\left(\bar{b}_i\right)}} \right)$$

$$= \Pr\left(\frac{\bar{b}_i - b'_i}{\sqrt{Var\left(\bar{b}_i\right)}} \ge \frac{\sum_{j=1}^{n} a'_{ij} x_j - b'_i}{\sqrt{Var\left(\bar{b}_i\right)}} \right) \ge \gamma_i \qquad (7)$$

$$i.e., 1 - \Pr\left(\frac{\bar{b}_i - b'_i}{\sqrt{Var\left(\bar{b}_i\right)}} \le \frac{\sum_{j=1}^{n} a'_{ij} x_j - b'_i}{\sqrt{Var\left(\bar{b}_i\right)}} \right) \ge \gamma_i$$

$$i.e., \Pr\left(\frac{\bar{b}_i - b'_i}{\sqrt{Var\left(\bar{b}_i\right)}} \le \frac{\sum_{j=1}^{n} a'_{ij} x_j - b'_i}{\sqrt{Var\left(\bar{b}_i\right)}} \right) \le 1 - \gamma_i$$

Let us consider t_i as $\phi\left(t_i\right) = 1 - \gamma_i$.

Then the constraints in equation (3) can be rewritten as

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$$\phi\left(\frac{\sum_{j=1}^{n} a'_{ij}x_j - b'_i}{\sqrt{Var\left(\bar{b}_i\right)}}\right) \le \phi\left(t_i\right),\ i = 1,2,\ldots,m$$

The above inequality will be satisfied if

$$\frac{\sum_{j=1}^{n} a'_{ij}x_j - b'_i}{\sqrt{Var\left(\bar{b}_i\right)}} \le t_i,\ i = 1,2,\ldots,m$$

Therefore, the probabilistic constraints (3) can be converted into deterministic constraints as:

$$\sum_{j=1}^{n} a'_{ij}x_j - b'_i - t_i\sqrt{Var\left(\bar{b}_i\right)} \le 0,\ i = 1,2,\ldots,m$$

Thus, we have obtained a deterministic model as follows:

Model 3

$$\max imise \quad Z_1(x) = \sum_{j=1}^{n} c'_{1j}x_j$$

$$\max imise \quad Z_2(x) = \sum_{j=1}^{n} c'_{2j}x_j$$

$$subject\ to \sum_{j=1}^{n} a'_{ij}x_j - b'_i - t_i\sqrt{Var\left(\bar{b}_i\right)} \le 0,\ i = 1,2,\ldots,m \qquad (8)$$

$$x_j \ge 0,\ \forall\ j$$

4 Computational algorithm

We expand the following two computational algorithms in fuzzy environment corresponding to the deterministic form of Model 3. Now the fuzzy programming technique for deriving a Stackelberg solution to the Model 3, which is summarised in the following way:

Algorithm I corresponding to the linear membership function

Step 4.1: Individually solve the upper and lower level objective functions with the constraints (8) and set the best $\left(Z_l^1\right)$ and worst $\left(Z_l^0\right)$ solution for each objective function, $l = 1,2$.

Step 4.2: Consider a linear membership function corresponding to each objective function which is defined as follows:

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$$\mu_l\left(Z_l(x)\right)=\begin{cases}1, & if\ Z_l(x)>Z_l^1,\\ \left(\dfrac{Z_l(x)-Z_l^0}{Z_l^1-Z_l^0}\right), & if\ Z_l^0\le Z_l(x)\le Z_l^1,\\ 0, & if\ Z_l(x)\le Z_l^0.\end{cases}$$

Also, consider a linear membership function for the upper-level decision variable $\left(x_1\right)$ with g tolerance value is interpreted by

$$\mu_{x_1}\left(x_1\right)=\begin{cases}\left(\dfrac{x_1-x_1^u+g}{g}\right), & if\ x_1^u-g\le x_1\le x_1^u,\\ \left(\dfrac{x_1^u+g-x_1}{g}\right), & if\ x_1^u\le x_1\le x_1^u+g,\\ 0, & if\ otherwise.\end{cases}$$

where x_1^u is the result set of the upper-level decision maker.

Step 4.3: In the manner of Zimmerman approach [26], solve the problem which is stated as below:

Model 4

$$\begin{aligned}&\text{max}imise && \beta\\ &subject\ to && \mu_1\left(Z_l(x)\right)\ge\beta,\quad \forall l\\ & && \mu_{x_1}\left(x_1\right)\ge\beta,\\ & && the\ equation\ (8),\\ & && x\ge 0\ and\ \beta\in[0,1].\end{aligned}$$

If the upper-level decision maker is fulfilled with the obtained best solution, then the solution becomes a satisfactory solution and go to **Step 4.4.** Otherwise, upper-level decision maker is restructured his/her membership function and then go to **Step 4.2.**

Step 4.4: Stop.

Algorithm II corresponding to the non-linear membership function

Repeat **Step 4.1** and go after the steps.

Step 4.2: Assume a non-linear membership function corresponding to each objective function and the upper-level decision variable $\left(x_1\right)$ with tolerance value g are specified by

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$$\mu_l\left(Z_l(x)\right)=\begin{cases}1, & if\ Z_l(x)>Z_l^1,\\ \left(\dfrac{e^{-r\left(\frac{Z_l(x)-Z_l^0}{Z_l^1-Z_l^0}\right)}-e^{-r}}{1-e^{-r}}\right), & if\ Z_l^0\le Z_l(x)\le Z_l^1,\\ 0, & if\ Z_l(x)\le Z_l^0.\end{cases}$$

$$\mu_{x_1}\left(x_1\right)=\begin{cases}\dfrac{e^{-r\left(\frac{x_1-x_1^u+g}{g}\right)}-e^{-r}}{1-e^{-r}}, & if\ x_1^u-g\le x_1\le x_1^u,\\ \dfrac{e^{-r\left(\frac{x_1^u-x_1+g}{g}\right)}-e^{-r}}{1-e^{-r}}, & if\ x_1^u\le x_1\le x_1^u+g,\\ 0, & if\ otherwise.\end{cases}$$

where x_1^u is the result set of the upper-level decision maker. Later than repeat **Step 4.3** to **Step 4.4.**

5. Numerical experiment

Here, we have present the effectiveness of the proposed computational algorithm using two numerical examples corresponding to trapezoidal and triangular fuzzy numbers respectively.

5.1 Example 1

Let us consider the following example:

Model 5

maximise $Z_1(x) = (15,17,20,22)x_1+(18,22,24,30)x_2+(26,30,33,34)x_3+(40,44,47,48)x_4$

maximise $Z_2(x) = (6,8,12,13)x_1+(13,14,17,19)x_2+(5,7,10,11)x_3+(12,14,16,22)x_4$

subject to
$$\begin{aligned}&(13,16,17,20)x_1+(30,33,37,40)x_2+(5,10,11,12)x_3+(65,69,78,90)x_4\le b_1,\\ &(12,14,16,20)x_1+(21,25,27,30)x_2+(50,53,61,64)x_3+(51,54,56,60)x_4\le b_2,\\ &(3,4,5,7)x_1+(25,26,28,30)x_2+(62,67,69,72)x_3+(23,25,28,32)x_4\ge b_3,\\ &(21,24,26,29)x_1+(24,27,32,35)x_2+(43,45,50,54)x_3+(52,54,57,60)x_4\ge b_4,\\ &x_j\ge 0,\ \forall j.\end{aligned}$$

To solve Model 5, we have considered mean and variance for random parameters b_1, b_2, b_3, b_4 which are given in Table 2.

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Table 2: Values of b_i corresponding to Example 1.

b_i	Mean	Variance
b_1	461	277
b_2	446	273
b_3	716	561
b_4	709	501

Using equations (5) and (8), Model 5 can be stated in the following model as:

Model 6

maximise $Z_1(x) = 18.5x_1 + 23.62x_2 + 30.64x_3 + 44.64x_4$

maximise $Z_2(x) = 9.73x_1 + 15.78x_2 + 8.22x_3 + 16.22x_4$

subject to $16.5x_1 + 35x_2 + 9.25x_3 + 75.81x_4 \leq 464.33,$ (9)

$15.6x_1 + 25.7x_2 + 57x_3 + 55.3x_4 \geq 449.30,$ (10)

$4.8x_1 + 27.29x_2 + 67.39x_3 + 27.08x_4 \geq 720.74,$ (11)

$25x_1 + 29.5x_2 + 48.06x_3 + 55.79x_4 \leq 713.48,$ (12)

$x_j \geq 0, \forall j.$

Using LINGO 15.0 iterative scheme, we first obtain the optimal solution for both level decision makers i.e., $Z_1^1 = 511.74$ at x_1=4.98, x_2=9.17, x_3=6.63, x_4=0.00; $Z_1^0 = 327.70$ at x_1=0.00, x_2=0.00, x_3=10.70, x_4=0.00 and $Z_2^1 = 247.60$ at x_1=4.98, x_2=9.17, x_3=6.63, x_4=0.00 and $Z_2^0 = 87.91$ at x_1=0.00, x_2=0.00, x_3=10.70, x_4=0.00. Now, with the help of algorithm I for linear membership function in Section 4, Model 6 is converted into a linear programming problem i.e., Model 7.

Model 7

maximise β

subject to $\left(\dfrac{Z_1(x) - 327.7}{184.04}\right) \geq \beta$

$\left(\dfrac{Z_2(x) - 87.91}{159.69}\right) \geq \beta$

$\left(\dfrac{x_1 + x_2 - 14.15 + g}{g}\right) \geq \beta$

the constraints (9)-(12),

$x_j \geq 0, \forall j.$

Now, using LINGO 15.0 iterative scheme and GA, Model 7 is solved for g = 0.1 and the optimum results are shown in the following table i.e., Table 3.

Table 3: Optimal results for Algorithm I in Example 1.

Method	B	$Z_1(x)$	$Z_2(x)$	x_1	x_2	x_3	x_4
LINGO 15.0	0.973	511.74	247.60	4.98	9.17	6.63	0.00
GA	0.685	513.55	243.78	3.67	9.13	6.72	0.54

Again, using algorithm II for non-linear membership function in Section 4, Model 6 is converted to a non-

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linear programming problem i.e., Model 8.

Model 8

maximise β

subject to
$$\frac{e^{-r\left(\frac{Z_1(x)-327.70}{184.04}\right)}-e^{-r}}{1-e^{-r}} \geq \beta,$$
$$\frac{e^{-r\left(\frac{Z_2(x)-87.91}{159.69}\right)}-e^{-r}}{1-e^{-r}} \geq \beta,$$
$$\frac{e^{-r\left(\frac{x_1+x_2-14.15+g}{g}\right)}-e^{-r}}{1-e^{-r}} \geq \beta,$$

the constraints (9)-(12),
$x_j \geq 0, \forall j.$

Using LINGO 15.0 iterative scheme and GA, Model 8 is solved for $r = 0.1$ and $g = 0.1$ and the optimum results are shown in Tables 4.

Table 4: Optimal results for Algorithm II.

Method	B	$Z_1(x)$	$Z_2(x)$	x_1	x_2	x_3	x_4
LINGO 15.0	0.99998	327.70	87.91	0.00	0.00	10.70	0.00
GA	0.89734	328.01	102.03	0.23	1.37	9.51	0.00

5.2 Example 2

Again, consider the following example:

Model 9

maximise $Z_1(x) = (202,300,390)x_1+(240,270,350)x_2+(180,200,230)x_3+(200,290,300)x_4$

maximise $Z_2(x) = (90,100,130)x_1+(115,120,130)x_2+(70,90,97)x_3+(80,85,100)x_4$

subject to
$$(35,40,52)x_1+(30,34,53)x_2+(40,41,50)x_3+(20,22,30)x_4 \leq b_1,$$
$$(25,40,42)x_1+(20,22,28)x_2+(40,45,50)x_3+(50,56,60)x_4 \leq b_2,$$
$$(40,42,45)x_1+(50,55,60)x_2+(60,64,70)x_3+(45,47,50)x_4 \geq b_3,$$
$$(33,40,45)x_1+(20,25,30)x_2+(26,29,30)x_3+(30,35,37)x_4 \geq b_4,$$
$$x_j \geq 0, \forall j.$$

Now, mean and variance for random parameters b_1, b_2, b_3, b_4 which are given in Table 5.

Table 5: Values of b_i corresponding to Example 1.

b_i	Mean	Variance
b_1	865	772
b_2	132	98
b_3	69	43
b_4	362	220

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Using equations (6) and (8), Model 9 can be rewritten in the following model as:

Model 10

$$\begin{aligned}
&\text{maximize } Z_1(x) = 297.33x_1 + 286.67x_2 + 203.33x_3 + 263.33x_4 \\
&\text{maximize } Z_2(x) = 106.67x_1 + 121.67x_2 + 85.67x_3 + 88.33x_4 \\
&\text{subject to } 42.33x_1 + 39x_2 + 43.67x_3 + 24x_4 \le 870.56, && (13)\\
&\qquad 35.67x_1 + 23.33x_2 + 45x_3 + 55.33x_4 \ge 133.98, && (14)\\
&\qquad 42.33x_1 + 55x_2 + 64.67x_3 + 47.33x_4 \ge 70.31, && (15)\\
&\qquad 39.33x_1 + 25x_2 + 28.33x_3 + 34x_4 \le 364.97, && (16)\\
&\qquad x_j \ge 0, \forall j.
\end{aligned}$$

Using LINGO 15.0 iterative scheme, we first obtain the optimal solution for both level decision makers i.e., $Z_1^1 = 4185.04$ at x_1=153.66, x_2=0.00, x_3=121.52, x_4=126.54; $Z_1^0 = 605.38$ at x_1=136.16, x_2=181.25, x_3=0.00, x_4=13.32 and $Z_2^1 = 1776.24$ at x_1=84.74, x_2=0.00, x_3=52.21, x_4=77.14 and $Z_2^0 = 213.89$ at x_1=0.00, x_2=0.00, x_3=0.00, x_4=2.42. Now, With the help of Algorithm I and II in Section 4, Model 10 is converted into two following models i.e., Model 11 and 12 respectively.

Model 11

$$\begin{aligned}
&\text{maximise } \beta \\
&\text{subject to } \left(\frac{Z_1(x) - 605.38}{3579.66}\right) \ge \beta \\
&\left(\frac{Z_2(x) - 213.42}{1562.82}\right) \ge \beta \\
&\left(\frac{x_1 + x_2 - 153.66 + g}{g}\right) \ge \beta \\
&\text{the constraints (13)-(16),} \\
&x_j \ge 0, \forall j.
\end{aligned}$$

Model 12

$$\begin{aligned}
&\text{maximise } \beta \\
&\text{subject to } \frac{e^{-r\left(\frac{Z_1(x)-605.38}{3579.66}\right)} - e^{-r}}{1 - e^{-r}} \ge \beta, \\
&\frac{e^{-r\left(\frac{Z_2(x)-213.42}{1562.82}\right)} - e^{-r}}{1 - e^{-r}} \ge \beta, \\
&\frac{e^{-r\left(\frac{x_1+x_2-153.66+g}{g}\right)} - e^{-r}}{1 - e^{-r}} \ge \beta, \\
&\text{the constraints (13)-(16),}
\end{aligned}$$

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$x_j \geq 0, \forall j.$

Using LINGO 15.0 iterative scheme and GA, Model 11 is solved for $r = 0.1$ and $g = 0.1$ and Model 12 is solved for $r = 0.1$ and $g = 0.2$. Subsequently optimum results are shown in Tables 6 and 7 respectively.

Table 6: Optimal results for Algorithm I in Example 2.

Method	β	g	$Z_1(x)$	$Z_2(x)$	x_1	x_2	x_3	x_4
LINGO 15.0	0.999998	0.1	4185.04	1776.24	0.00	14.599	0.00	0.00
GA	0.879961	0.1	4191.82	1760.89	0.73	12.45	1.84	0.12

Table 7: Optimal results for Algorithm II in Example 2.

Method	β	g	r	$Z_1(x)$	$Z_2(x)$	x_1	x_2	x_3	x_4
LINGO 15.0	0.99996	0.2	0.1	2697.39	1025.44	3.13	2.68	1.76	2.44
GA	0.88967	0.2	0.1	2700.61	1025.68	3.43	2.86	1.32	2.25

6. Results and discussion

In this section, we present the discussion on the experimentation results performed on the proposed mathematical model under this study. In Example 1, from Tables 2 and 3, it is observed that the optimal solutions are obtained using LINGO 15.0 iterative scheme and GA respectively. In case of linear membership function under GA, the best solution is reached for $\lambda = 0.685$ and $d = 0.1$ corresponding to both level decision makers $Z_1 = 513.55$ and $Z_2 = 243.78$. Again, for the non-linear membership function under GA, the results obtained $Z_1 = 328.01$ and $Z_2 = 102.03$ with $\lambda = 0.89734$, $r = 0.1$ and g = 0.1 respectively. In Example 2, from Tables 5 and 6, it is accomplished that the optimal solution are obtained in the case of linear membership function under GA are $Z_1 = 4191.82$, $Z_2 = 1760.89$ with $\lambda = 0.879961$ and $g = 0.2$. So the optimal solution occurred corresponding to the linear membership function under GA. Lastly, we concluded that the two numerical experiments are attained the best solution corresponding to only consideration of linear membership function under GA.

7. Concluding remarks and outlook

In this paper, an effort has been made to solve a BLPP for Stackelberg game where the cost parameters and the constraint parameters are fuzzy numbers except right hand side of the constraint parameters may be triangular or trapezoidal whereas the right side of the constraint follow a normal distribution. Based on Mellin's transformation , a fuzzy number is converted into crisp value. After that, using stochastic programming procedure, all probabilistic constraints have been transformed into a deterministic form. Then based on computational algorithm reduced problem solved by LINGO 15.0 iterative scheme and GA respectively and compared the results.

The major four aspects are treated as new features of the paper as follows

(a) We have a proposed probabilistic BLPP in Stackelberg game under fuzzy environment using liner and non-linear membership function.

(b) All triangular or trapezoidal fuzzy numbers are converted into crisp value using Mellin's transformation.

(c) Based on stochastic programming approach, all probabilistic constraints which can be followed a normal distribution to convert into a deterministic form.

(d) The results are shown in comparatively by LINGO 15.0 iterative scheme and GA corresponding to linear and non-linear membership functions respectively.

In future, this method can be studied with some other type of non-linear membership functions like parabolic, hyperbolic etc. However, it is expected that the computational algorithm obtainable here may open a lot of new vistas of future works including managerial decision making problem, transportation problem in supply-chain network etc.

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