An Analysis of Integral Equations and Their use to Boundary Value Problem

Neelima Soni[1],Dr. Syed Shahnawaz Ali[2]

Research Scholar, Department of Mathematics, SSSUTMS, Sehore[1]
Professor, Department of Mathematics, SSSUTMS, Sehore[2]

Abstract

Integral Equations are among the most effective methods in many areas of pure analysis, including theories of functional analysis and stochastic processes. In many areas of mechanics and mathematical physics, it is one of the most important branches of mathematical analysis. We shall discuss the integral equations in numerous physical concerns and their applications in this research. They also relate to issues with mechanical vibration, analytical function theory, orthogonal systems, and infinitely many variable quadratic form theory. In this article, we discuss how integral equations can be used to solve boundary value problems.

Keyword:

Integral Equation, Boundary value problem, Orthogonal system, Functional analysis.

1. Introduction:

This paper is concerned with the boundary integral equation method for a problem in potential theory, namely the Dirichlet boundary value problem in a non-locally perturbed halfplane. The main aim of the paper is to discuss the well-posedness of this problem and of a novel second kind boundary integral equation formulation. Our motivation for studying this problem is that it arises in the theory of classical free surface water wave problems, for which boundary integral equation methods are well-established as a computational and theoretical tool [3, 4, 8, 13]. In particular, accurate numerical schemes, based on boundary integral equation formulations, for the time dependent water wave problem have been proposed and fully analyzed in [3, 4, 13], these papers providing a full nonlinear stability analysis for the spatial discretizations they propose. A significant component in this analysis is the well-posedness of the boundary integral equation formulation. A key restriction in the analysis in the above papers is the requirement that the free surface be periodic (in 2D) or doubly-periodic (in 3D). This restriction is helpful theoretically and computationally. It enables the boundary integral equation on the free surface to be reduced to one on a (bounded) single periodic cell, which can then be discretized with a finite size mesh. Moreover, the boundary integral formulation is of second kind with a compact integral operator, and therefore standard Riesz/Fredholm theory gives well-posedness via the Fredholm alternative. As a step towards a broader extension of the results of [3, 4, 13], this paper is concerned with studying the Dirichlet boundary value problem for Laplace's equation in a perturbed half-plane Ω without the requirement that the boundary $\partial\Omega$ of Ω be periodic. To simplify our task somewhat, we impose other conditions on the boundary, namely: the boundary surface is the graph of a bounded continuous function (this excludes configurations relevant to breaking waves); the surface is sufficiently smooth (at least Lyapunov, that is, the unit normal direction is H¨older continuous). We note that there exists a well-developed L2 theory of the boundary integral equation method for the

Dirichlet problem when the boundary is the graph of an arbitrary Lipschitz function and the Dirichlet data is in L2(∂Ω), see e.g., Verchota [19], Meyer and Coifman [15] and the references therein. However, this theory does not extend to the case of data in L∞(∂Ω). In particular, even if the boundary is smoother than Lipschitz, the standard double layer potential operator is not welldefined on L∞(∂Ω). Addressing this difficulty, our aim in this paper is to develop a theory which includes the case when neither the surface elevation nor the Dirichlet data exhibit decay at infinity. To obtain a boundary integral equation formulation appropriate to this case we make the ansatz that the solution can be represented as a double layer potential supported on the (infinite) boundary of the domain, with the twist that we replace the standard fundamental solution of Laplace's equation in 2D with the Dirichlet Green's function for a half-plane ΩH that strictly includes the domain Ω. We note that the case of periodic surface elevation and Dirichlet data will be included as a special case in the theory we develop.

2. Boundary Value Problem

A **Boundary value problem** is a system of ordinary differential equations with solution and derivative values specified at more than one point. Most commonly, the solution and derivatives are specified at just two points (the boundaries) defining a two-point boundary value problem.
A two-point boundary value problem (BVP) of total order n on a finite interval $[a,b]$ may be written as an explicit first order system of ordinary differential equations (ODEs) with boundary values evaluated at two points as

$$y'(x)=f(x,y(x)),x\in(a,b),g(y(a),y(b))=0(1)$$

Here, $y,f,g\in Rn$ and the system is called explicit because the derivative y' appears explicitly. The n boundary conditions defined by g must be independent; that is, they cannot be expressed in terms of each other (if g is linear the boundary conditions must be linearly independent).

In practice, most BVPs do not arise directly in the form (1) but instead as a combination of equations defining various orders of derivatives of the variables which sum to n . In an explicit BVP system, the boundary conditions and the right hand sides of the ordinary differential equations (ODEs) can involve the derivatives of each solution variable up to an order one less than the highest derivative of that variable appearing on the left hand side of the ODE defining the variable. To write a general system of ODEs of different orders in the form (1), we can define y as a vector made up of all the solution variables and their derivatives up to one less than the highest derivative of each variable, then add trivial ODEs to define these derivatives. See the section on initial value problems for an example of how this is achieved. See also Ascher et al.(1995) who show techniques for rewriting boundary value problems of various orders as first order systems. Such rewritten systems may not be unique and do not necessarily provide the most efficient approach for computational solution.

The words *two-point* refer to the fact that the boundary condition function g is evaluated at the solution at the two interval endpoints a and b unlike for initial value problems (IVPs) where the n initial conditions are all evaluated at a single point. Occasionally, problems arise where the function g is also evaluated at the solution at other points in (a,b) . In these cases, we have a multipoint BVP. As shown in Ascher et al. (1995), a multipoint problem may be converted to a two-point problem by defining separate sets of variables for each subinterval between the points and adding boundary conditions which ensure continuity of the variables across the whole interval. Like rewriting the original BVP in the compact form (1), rewriting a multipoint problem as a two-point problem may not lead to a problem with the most efficient computational solution.

Most practically arising two-point BVPs have separated boundary conditions where the function g may be split into two parts (one for each endpoint):

$$ga(y(a))=0, gb(y(b))=0.$$

Here, $ga \in Rs$ and $gb \in Rn-s$ for some value s with $1<s<n$ and where each of the vector functions ga and gb are independent. However, there are well-known, commonly arising, boundary conditions which are not separated; for example, consider periodic boundary conditions which, for a problem written in the form of equation (1), are

$$y(a)-y(b)=0.$$

Existence and uniqueness

Questions of existence and uniqueness for BVPs are much more difficult than for IVPs. Indeed, there is no general theory. However, there is a vast literature on individual cases; see Bernfeld and Lakshmikantham (1974) for a survey of a variety of techniques that may be used. Consider the IVP

$$y'(x)=f(x,y(x)), y(a)=s (2)$$

corresponding to the ODE in (1). If this IVP has a solution for all choices of initial vectors s then the existence of a solution to (1) hinges on the solvability of the nonlinear system of equations

$$g(s,y(b;s))=0 (3)$$

where $y(b;s)$ is the solution of the IVP (2) evaluated at $x=b$ for the initial value $y(a)=s$. If there is a solution then it is the unique solution (among solutions of this type) if the nonlinear system $g(s,y(b;s))=0$ has just one solution s .

For linear BVPs, where the ODEs and boundary conditions are both linear, the equation $g(s,y(b;s))=0$ is a linear system of algebraic equations. Hence, generally there will be none, one or an infinite number of solutions, analogously to the situation with systems of linear algebraic equations.

In addition to the possibilities for linear problems, nonlinear problems can also have a finite number of solutions. Consider the following simple model of the motion of a projectile with air resistance:

$$y'v'\phi'===\tan(\phi),-gv\tan(\phi)-vv\sec(\phi),-gv2.(4)$$

These equations may be viewed as describing the planar motion of a projectile fired from a cannon. Here, y is the height of the projectile above the level of the cannon, v is the velocity of the projectile, and ϕ is the angle of the trajectory of the projectile with the horizontal. The independent variable x measures the horizontal distance from the cannon. The constant v represents air resistance (friction) and g is the appropriately scaled gravitational constant. This model neglects three–dimensional effects such as cross winds and the rotation of the projectile. The initial height is $y(0)=0$ and the muzzle velocity $v(0)$ for the cannon is fixed. The standard projectile problem is to choose the initial angle of the cannon and hence of the projectile, $\phi(0)$, so that the projectile will hit a target at the same height as the cannon at a distance $x=xend$; that is, we require $y(xend)=0$. Altogether the boundary conditions are

$$y(0)=y(xend)=0, v(0) \text{given}.$$

ISSN- 2394-5125 VOL 07, ISSUE 19, 2020

Does this BVP have a solution? Physical intuition suggests that it certainly does not for *xend* beyond the range of the cannon for the fixed muzzle velocity $v(0)$. On the other hand, if *xend* is small enough, we do expect a solution, but is there only one? To see that there is not, consider the case when the target is very close to the cannon. We can hit the target by shooting with an almost flat trajectory or by shooting high and dropping the projectile mortar-like on the target. That is, there are (at least) two solutions that correspond to initial angles $\phi(0)=\phi low>0$ and $\phi(0)=\phi high<\pi/2$.

3. Integral Equation

An integral equation is defined as an equation in which the unknown function u(x) to be determined appear under the integral sign. The subject of integral equations is one of the most useful mathematical tools in both pure and applied mathematics. It has enormous applications in many physical problems. Many initial and boundary value problems associated with ordinary differential equation (ODE) and partial differential equation (PDE) can be transformed into problems of solving some approximate integral equations (Refs. [2], [3] and [6]). The development of science has led to the formation of many physical laws, which, when restated in mathematical form, often appear as differential equations. Engineering problems can be mathematically described by differential equations, and thus differential equations play very important roles in the solution of practical problems. For example, Newton's law, stating that the rate of change of the momentum of a particle is equal to the force acting on it, can be translated into mathematical language as a differential equation. Similarly, problems arising in electric circuits, chemical kinetics, and transfer of heat in a medium can all be represented mathematically as differential equations. A typical form of an integral equation in u(x) is of the form

$$u(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} K(x,t)u(t)dt$$

Where K(x, t) is called the kernel of the integral equation, and α(x) and β(x) are the limits of integration. It can be easily observed that the unknown function u(x) appears under the integral sign. It is to be noted here that both the kernel K(x, t) and the function f (x) in equation are given functions; and λ is a constant parameter. The prime objective of this text is to determine the unknown function u(x) that will satisfy equation using a number of solution techniques. We shall devote considerable efforts in exploring these methods to find solutions of the unknown function.

Classification of integral equations

An integral equation can be classified as a linear or nonlinear integral equation as we have seen in the ordinary and partial differential equations. In the previous section, we have noticed that the differential equation can be equivalently represented by the integral equation. Therefore, there is a good relationship between these two equations. The most frequently used integral equations fall under two major classes, namely Volterra and Fredholm integral equations. Of course, we have to classify them as homogeneous or nonhomogeneous; and also, linear or nonlinear. In some practical problems, we come across singular equations also. In this text, we shall distinguish four major types of integral equations – the two main classes and two related types of integral equations. In particular, the four types are given below:

- Volterra integral equations
- Fredholm integral equations

1. Volterra integral equations

The most standard form of Volterra linear integral equations is of the form

$$\phi(x)u(x) = f(x) + \lambda \int_a^x K(x,t)u(t)dt$$

Where the limits of integration are function of x and the unknown function u(x) appears linearly under the integral sign. If the function φ(x) = 1, then equation simply becomes

$$u(x) = f(x) + \lambda \int_a^x K(x,t)u(t)dt$$

and this equation is known as the Volterra integral equation of the second kind; whereas if φ(x) = 0, then equation becomes

$$f(x) + \lambda \int_a^x K(x,t)u(t)dt = 0$$

which is known as the Volterra equation of the first kind.

2. Fredholm integral equations

The most standard form of Fredholm linear integral equations is given by the form

$$\phi(x)u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt$$

where the limits of integration a and b are constants and the unknown function u(x) appears linearly under the integral sign. If the function φ(x) = 1, then becomes simply

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt$$

and this equation is called Fredholm integral equation of second kind; whereas if φ(x) = 0, then (1.11) yields

$$f(x) + \lambda \int_a^b K(x,t)u(t)dt = 0$$

which is called Fredholm integral equation of the first kind

It is important to note that integral equations arise in engineering, physics, chemistry, and biological problems. Many initial and boundary value problems associated with the ordinary and partial differential equations can be cast into the integral equations of Volterra and Fredholm types, respectively. If the unknown function u(x) appearing under the integral sign is given in the functional form F(u(x)) such as the power of u(x) is no longer unity, e.g. F(u(x)) = un(x), n = 1, or sin u(x) etc., then the Volterra and Fredholm integral equations are classified as nonlinear integral equations.

4. Preliminies

Theorem (Open mapping theorem)

Suppose that T : X → Y is a one-to-one, onto and bounded linear map between Banach spaces X and Y . Then T −1 : Y → X is bounded.

Therefore, both µI − K and (µI − K) −1 are one-to-one, onto and bounded linear operators. The spectrum of a bounded linear operator K on an infinite-dimensional space is divided into three cases:

- The point spectrum of K consists of all $\mu \in \sigma(K)$ such that $\mu I - K$ is not one-to-one. The point spectrum of K is known as the eigenvalue set of K.
- The continuous spectrum of K consists of all $\mu \in \sigma(K)$ such that $\mu I - K$ is one-to-one but not onto, and has dense range.
- The residual spectrum of K consists of all $\mu \in \sigma(K)$ such that $\mu I - K$ is one-to-one but not onto, and does not have dense range.

Next, we state four Fredholm's theorem which can be found, e.g. in (Mikhlin (1957), Atkinson (1997)). Let the kernel $k(x, y) : [a, b] \times [a, b] \to C$ be a continuous function. Let also the solutions $\rho(y)$ and $g(y)$ belong to the space $C^0[a, b]$ of continuous functions defined in a closed interval $a \le x \le b$ with norm $||\rho|| = \max_{x \in J} |\rho(x)|$ where $J = [a, b]$. Then, the following holds:

Theorem (Fredholm's second theorem)
To each characteristic value there corresponds at least one eigenfunction. The number of linearly independent eigenfunctions
$\rho_1(y), \rho_2(y), ..., \rho_n(y)$,
corresponding to a given characteristic value, is finite.

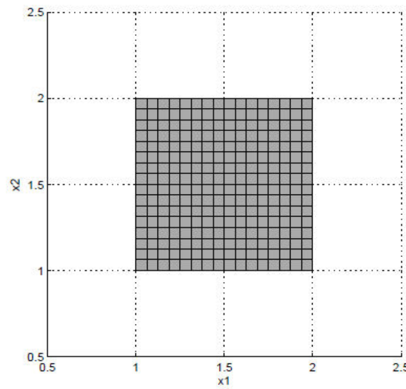
4.1 Numerical Examples of BDIE for Neumann Problem

For the numerical experiments, we solve the BDIE for Neumann problem (3.10) on several two-dimensional test domains such as square, circular and parallelogram domains. These test domains will also be used in the next chapters that deals with solving BDIE for Dirichlet problem and Localized-Boundary Domain Integral equation (LBDIEs) for Neumann and Dirichlet problems. The first test domain that we consider is a square $1 < x_1 < 2$, $1 < x_2 < 2$. The second test domain is a circular domain with centre (2, 2) and unit radius. The final test domain is

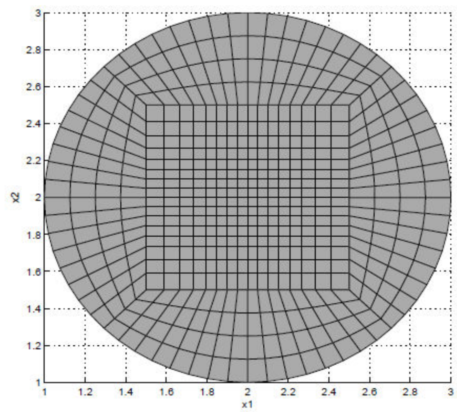
a parallelogram domain with vertices (3, 1), (4, 1), (6, 2) and (5, 2). Figure shows some meshes examples for the three test domains where J is the number of nodes. In the numerical experiments, we consider several interior Neumann problems with the following parameters:

1. $a(x) = 1$, $f(x) = 0$ for $x \in \Omega \cup \partial\Omega$, with $t(x) = \nu_1(x)$, $x \in \partial\Omega$,
2. $a(x) = x_2^2$, $f(x) = 0$ for $x \in \Omega \cup \partial\Omega$, with $t(x) = x_2^2 \nu_1(x)$, $x \in \partial\Omega$,
3. $a(x) = x_2^4$, $f(x) = 0$ for $x \in \Omega \cup \partial\Omega$, with $t(x) = x_2^4 \nu_1(x)$, $x \in \partial\Omega$,
4. $a(x) = x_2^6$, $f(x) = 0$ for $x \in \Omega \cup \partial\Omega$, with $t(x) = x_2^6 \nu_1(x)$, $x \in \partial\Omega$,
5. $a(x) = x_2^8$, $f(x) = 0$ for $x \in \Omega \cup \partial\Omega$, with $t(x) = x_2^8 \nu_1(x)$, $x \in \partial\Omega$,
6. $a(x) = x_2^{10}$, $f(x) = 0$ for $x \in \Omega \cup \partial\Omega$, with $t(x) = x_2^{10} \nu_1(x)$, $x \in \partial\Omega$,
7. $a(x) = x_2^2$, $f(x) = 2x_2^2$ for $x \in \Omega \cup \partial\Omega$, with $t(x) = 2x_1 x_2^2 \nu_1(x)$, $x \in \partial\Omega$,

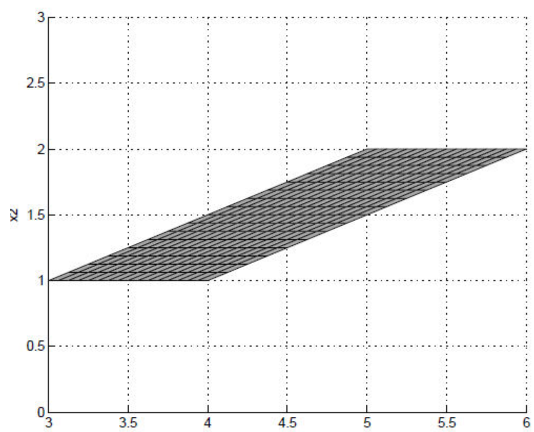
The exact solutions for Neumann problem in Tests 1-6 and Test 7 respectively, $u(x) = x_1$, $x \in \Omega \cup \partial\Omega$, $u(x) = x_1^2$, $x \in \Omega \cup \partial\Omega$. All numerical computations are done using Fortran package written by the author, with the double precision. We solve the linear system by two approaches. The first one is using LU decomposition and the second is using the Neumann series.



(a) Square domain with $J = 289$.



(b) The unit-radius circular domain with $J = 545$.



(c) The parallelogram domain with $J = 289$.

Figure 1: Test domains

For each domain, we present a posteriori relative errors for the approximate solution

$$\epsilon(u) = \frac{\max\limits_{1\le j\le J} \left|u_{approx}(x^j) - u_{exact}(x^j)\right|}{\max\limits_{1\le j\le J} |u_{exact}(x^j)|},$$

and for its gradient

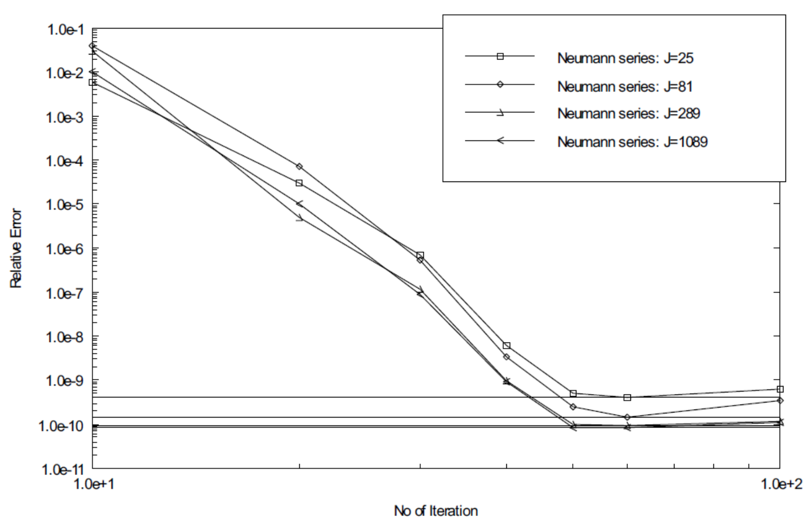
$$\epsilon(\nabla u) = \frac{\max\limits_{1\le m\le M} |\nabla u_{approx}(x_c^m) - \nabla u_{exact}(x_c^m)|}{\max\limits_{1\le m\le M} |\nabla u_{exact}(x_c^m)|},$$

where x m c are centres of the quadrilateral domain elements em. We determine ∂uapprox/∂x1 and ∂uapprox/∂x2 at the middle of each interior domain element. The numerical results of ∂u/∂x1 and ∂u/∂x2 are based on the following interpolation:

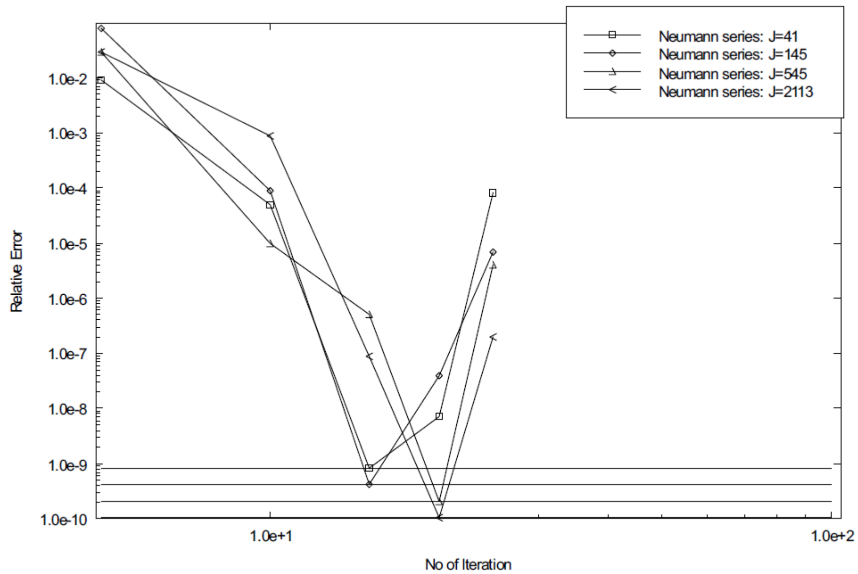
$$\begin{aligned}\frac{\partial u}{\partial x_1} &= \sum_j \frac{\partial \phi_j(x)}{\partial x_1} u(x^j),\\ \frac{\partial u}{\partial x_2} &= \sum_j \frac{\partial \phi_j(x)}{\partial x_2} u(x^j), \qquad x,\ x^j \in \Omega \cup \partial\Omega.\end{aligned}$$

The interpolation formulas in the local coordinates are given as follows:

$$\begin{aligned}\frac{\partial u}{\partial x_1} &= \sum_{N=1}^{4}\sum_{j=1}^{2} \frac{\partial \Phi_N(\xi(x))}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_1} u(x_N^m),\\ \frac{\partial u}{\partial x_2} &= \sum_{N=1}^{4}\sum_{j=1}^{2} \frac{\partial \Phi_N(\xi(x))}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_2} u(x_N^m), \quad x,\ x_N^m \in \Omega \cup \partial\Omega.\end{aligned}$$



(a) Square



(b) Circular domain

Conclusion

Reference

[1] P. Amodio, J.R. Cash, G. Roussos, R.W. Wright, G. Fairweather, I. Gladwell, G.L. Kraut and M. Paprzycki, Almost block diagonal linear systems: sequential and parallel solution techniques, and applications, *Numer. Lin. Algebra Applics.* **7** (2000) 275-317.

[2] U.M. Ascher, R.M.M. Mattheij, R.D. Russell, *Numerical Solution of Boundary Value Problems for Ordinary Differential Equations*, SIAM Classics in Applied Mathematics 13, SIAM, Philadelphia, PA, 1995.

[3] U.M. Ascher and L.R. Petzold, *Computer Methods for Ordinary Differential Equations and Differential Algebraic Equations*, SIAM, Philadelphia, PA, 1998.

[4] S.R. Bernfeld and V. Lakshmikantham, *An Introduction to Nonlinear Boundary Value Problems*, Mathematics in Science and Engineering 109, Academic Press, New York, NY, 1974.

[5] H.B. Keller, *Numerical Methods for Two-Point Boundary-Value Problems*, Dover, New York, NY, 1992.

[6] R.M.M. Mattheij and J. Molenaar, *Ordinary Differential Equations in Theory and Practice*, SIAM Classics in Applied Mathematics 43, SIAM, Philadelphia, PA, 2002.

[7] J.D. Pryce, *Numerical Solution of Sturm-Liouville Problems*, Clarendon Press, Oxford, UK, 1993.

[8] L.F. Shampine, I. Gladwell and S. Thompson, *Solving ODEs with MATLAB*, Cambridge University Press, Cambridge, UK, 2003.

[9] A. Zettl, *Sturm-Liouville Theory*, Mathematical Surveys and Monographs 121, AMS, Providence, RI, 2005.

[10] M.H.Aliabadi. *TheBoundaryElementMethod,Vol.2,ApplicationsinSolidsandStructures*.Wiley,Chichester,2002.

[11] M. Ameen.*Computational Elasticity: Theory of Elasticity, Finite and Boundary ElementMethods*.AlphaScienceInternationalLtd,Harrow,UK,2005.

[12] R.A. Archer and R.N. Horne.Flow simulation in heterogeneous reservoirs using the dualreciprocity boundary element method and the Green element method.In *ECMORVIConference*,Peebles,Scotland,1998.

[13] R.A. Archer and R.N. Horne. The Green element method for numerical well test analysis.In *SPE Annual Technical Conference and Exhibition*, Dallas, Texas, 1–4 October 2000.SocietyofPetroleumEngineers.

[14] R.A.ArcherandR.N.Horne.Greenelementmethodandsingularityprogrammingfornumerical well test analysis. *Engineering Analysis with Boundary Elements*, 26(6):537–546,2002.

[15] R.A.Archer,R.N.Horne,andO.Onyejekwe.Petroleumreservoirengineeringapplicationsof the dual reciprocity boundary element method and the Green element method. In *21stWorld Conference on the Boundary Element Method*, Oxford University, 25–27 August1999.