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Some Chaotic Properties of $\mathbb{G}-$ periodic Shadowing Property

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Abstract:
Let $(\mathcal{M}, d)$ be a metric $\mathbb{G}-$space and $\phi : \mathcal{M} \to \mathcal{M}$ be a continuous map. This paper aims to study the idea of the $\mathbb{G}-$periodic shadowing property ($\mathbb{G}$ Per.SP) for a continuous map on $\mathbb{G}-$space and achieves the relative of the $\mathbb{G}$ Per.SP with $\mathbb{G}-$shadowing property ($\mathbb{G}$ SP). Also, if ϕ has the $\mathbb{G}$ Per.SP, then ϕ^n has the $\mathbb{G}$ Per.SP for every $n \in \mathbb{N}$. We show that if ϕ is a $\mathbb{G}$-expansive and has the $\mathbb{G}$ SP then ϕ has the $\mathbb{G}$ Per.SP, and if the map ϕ on compact metric $\mathbb{G}-$space has $\mathbb{G}-$chain transitive and the $\mathbb{G}$ Per.SP, then ϕ has the $\mathbb{G}$ SP with $\mathbb{G}-$transitivity. We show that the map ϕ on compact metric $\mathbb{G}-$space, ϕ is a $\mathbb{G}$–expansive and $\mathbb{G}-$chain mixing, if ϕ^n has $\mathbb{G}$ Per.SP for some $n \in \mathbb{N}$,such that $n \neq 1$ then ϕ has $\mathbb{G}$ Per.SP. Moreover, we prove that if a map ϕ be pseudo-equivariant with dense set of $\mathbb{G}_\phi$ −periodic points which has the $\mathbb{G}$ Per.SP and $\mathbb{G}-$average shadowing property($\mathbb{G}$ASP) then ϕ is $\mathbb{G}-$chain mixing. Finally, we show that if $(\mathcal{M}, d)$ is a compact metric $\mathbb{G}-$space having two points at least, ϕ be a $\mathbb{G}-$distal homeomorphism and ϕ is $\mathbb{G}-$chain mixing , then ϕ does not have the $\mathbb{G}$ Per.SP.

Keywords: $\mathbb{G}-$shadowing; periodic shadowing; $\mathbb{G}-$periodic shadowing; $\mathbb{G}$–expansive; The $\mathbb{G}-$average shadowing; $\mathbb{G}-$transitivity ; topologically $\mathbb{G}-$chain mixing.

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Introduction

The shadowing property in the theory of dynamical systems, is one of the effective concepts. The idea of shadowing property (SP) was introduced by Anosov in 1967 [1]. The average shadowing property (ASP) was showed by Blank in study of chaotic dynamical systems[2].The idea of periodic shadowing property (Per.SP) was presented by Peter E. Kloeden and K. T. Palmer[3]. A. Darabi and F. Forouzanfar studied the Per.SP and they showed that expansive dynamical systems having the pseudo-trajectory tracing properly (PTTP) also have the Per.SP [4].
A. Darabi introduced some properties of the Per.SP and he showed that continuous maps on a compact metric space with the Per.SP and the ASP have the SP, too [5]. The notion of $\mathbb{G}-$space was introduced by R. S. Palais in 1960 [6]. The notion of the $\mathbb{G}-$pseudo trajectory tracing property on a metric $\mathbb{G}-$space ($\mathbb{G}-$PTTP) was introduced by Ekta Shah and T. K. Das. They studied various examples and properties of maps and they got features for the identity map that possesses $\mathbb{G}-$PTTP [7]. The $\mathbb{G}$ SP was presented by Ekta Shah for map ϕ and through examples she observed that $\mathbb{G}-$shadowing depends on the action of a group $\mathbb{G}$ acting on $\mathcal{M}$, she also defined concept of the periodic points on a metric $\mathbb{G}-$space ($\mathbb{G}-$periodic points) and she studied the behavior of the set of $\mathbb{G}-$periodic points of a positively $\mathbb{G}-$expansive map having the $\mathbb{G}$ SP [8].
The $\mathbb{G}$ Per.SP for maps on $\mathbb{G}-$spaces is studied in this paper and we prove some the similar results on the Per.SP in the metric space with some the chaotic properties and we put sufficient conditions to prove these results on $\mathbb{G}-$spaces.

Prelusions

Let $\mathbb{N}$ be symbolizes the set of natural numbers, $\mathcal{N}_0 = \{0\} \cup \mathbb{N}$ and $\mathbb{Z}$ symbolizes the set of integers numbers. A *topological group* is a triple $(\mathbb{G}, \mathcal{T}, *)$, where $(\mathbb{G}, *)$ is a group and $\mathcal{T}$ is a Hausdorff topology on $\mathbb{G}$ such that the map $\phi: \mathbb{G} \times \mathbb{G} \to \mathbb{G}$ defined by $\phi(m, y) = my^{-1}$ is continuous. By a $\mathbb{G}-$space , we mean a triple $(\mathcal{M}, \mathbb{G}, \theta)$, where $\mathcal{M}$ is a Hausdorff space, $\mathbb{G}$ is a topological group and $\theta: \mathbb{G} \times \mathcal{M} \to \mathcal{M}$ is a continuous action of $\mathbb{G}$ on $\mathcal{M}$ satisfying $\theta(e, m) = m$ and $\theta(g_1, \theta(g_2, m)) = \theta(g_1 g_2, m)$, where e is the identity of $\mathbb{G}$, $m \in \mathcal{M}$ and $g_1, g_2 \in \mathbb{G}$. An action θ of $\mathbb{G}$ on $\mathcal{M}$ is called trivial if $\theta(g, m) = m$ for each g in $\mathbb{G}$ and m in $\mathcal{M}$.For $m \in \mathcal{M}$, the set $\mathbb{G}(m) = \{\theta(g, m): g \in \mathbb{G}\}$, is called the $\mathbb{G}-$trajectory of m in $\mathcal{M}$. We will be denoted of $\theta(g, m)$ by gm. If $\mathcal{M}, Y$ are $\mathbb{G}-$spaces, then a continuous map $h: \mathcal{M} \to Y$ is named equivariant map if $h(gm) = gh(m)$ for each g in $\mathbb{G}$ and each m in $\mathcal{M}$, in case an equivariant map is a homeomorphism, then h^{-1} is also equivariant. If h is such that for each $m \in \mathcal{M}$, $h(\mathbb{G}(m)) = \mathbb{G}(h(m))$ then h is said to be pseudo-equivariant. Clearly every equivariant map is a pseudo-equivariant map but the converse need not be true [9].
In the following section, we introduce some definitions that we need them in this paper and we recall essential definitions. We symbolizes to metric $\mathbb{G}-$ space on which a topological group $\mathbb{G}$ with metric d by $(\mathcal{M}, d)$ and the map $\phi : \mathcal{M} \to \mathcal{M}$, we mean $\phi: (\mathcal{M}, d) \to (\mathcal{M}, d)$. By $(\mathcal{M}, d)$ be a compact metric $\mathbb{G}-$ space, we mean a compact metric $\mathbb{G}-$ space on that a compact topological group $\mathbb{G}$ with metric d.

Definition 2.1 [10]
Let $(\mathcal{M}, d)$ be a compact metric space and $\phi : \mathcal{M} \to \mathcal{M}$ be continuous map. A sequence $\{m_i, i \in \mathbb{Z}\}$ is called trajectory of ϕ if $\forall i \in \mathbb{Z}$ we have $m_{i+1} = \phi(m_i)$ and we named it a δ -pseudo - trajectory of ϕ, $\forall i \in \mathbb{Z}$ we have $d(\phi(m_i), m_{i+1}) \le \delta$, the map ϕ has the shadowing property (SP) if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that every δ-pseudo-trajectory $\{m_i, i \in \mathbb{Z}\}$ is ε- shadowed by the trajectory $\{\phi^i(m), i \in \mathbb{Z}\}$ for some $z \in \mathcal{M}$, that is, $\forall i \in \mathbb{Z}$ we have $d(\phi^i(z), m_i) \le \varepsilon$.
A sequence $\{m_i, i \in \mathbb{Z}\}$ in $\mathcal{M}$ is named a δ - average pseudo-trajectory of $\mathcal{M}$ if $\exists\, N > 0$, and $N = N(\delta)$ such that $\forall\, n \ge N$ and $k \in \mathbb{N}$ we have

$$\frac{1}{n}\sum_{i=0}^{n-1} d(\phi(m_{i+k}), m_{i+k+1}) < \delta ,$$

we say that ϕ has the ASP if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that every δ −average pseudo- trajectory $\{m_i, i \in \mathbb{Z}\}$ is ε - shadowed in average by the trajectory of some point $z \in \mathcal{M}$, it means

$$\lim_{n\to\infty} \sup \frac{1}{n}\sum_{i=0}^{n-1} d(\phi^i(z), m_i) < \varepsilon.$$

Definition 2.2 [8]
Let $(\mathcal{M}, d)$ be a metric $\mathbb{G}-$space and $\phi : \mathcal{M} \to \mathcal{M}$ be a continuous map. For a real number δ, that is positive, a sequence of points $\{m_i : a < i < b\}$ in $\mathcal{M}$ is called $(\delta, \mathbb{G})$ −pseudo-trajectory for ϕ if $\forall\, i,\ a < i < b-1$, $\exists\, g_i \in \mathbb{G}$ such that $d(g_i\, \phi(m_i), m_{i+1}) < \delta$.
For a given $\varepsilon > 0$, a $(\delta, \mathbb{G})$ −pseudo trajectory $\{m_i : a < i < b\}$ for ϕ is said to ε −shadowed with a point m of $\mathcal{M}$ if $\forall i, a < i < b$, $\exists\, p_i \in \mathbb{G}$ such that $d(\phi^i(m), p_i m_i) < \varepsilon$, the map ϕ has the $\mathbb{G}$ SP if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that each $(\delta, \mathbb{G})$ −pseudo-trajectory for ϕ is ε −shadowed with a point of $\mathcal{M}$. Note that if ϕ is bijective we take $-\infty < a < b < \infty$, also, when ϕ is not bijective we take $0 \le a < b < \infty$.

Definition 2.3 [11]

Let $(\mathcal{M}, ɗ)$ be a metric $\mathbb{G}-$space and $\phi:\mathcal{M}\to\mathcal{M}$ be a continuous map. A point $m\in\mathcal{M}$ is named $\mathbb{G}-$*periodic point* of ϕ if $\exists\, n\in\mathbb{Z},\ n>0$ and $g\in\mathbb{G}$ such that $g\phi^{n}(m)=m$. The set of $\mathbb{G}-$periodic points is symbolized by $P_{\mathbb{G}}(\phi)$.

Definition 2.4 [12]

Let $(\mathcal{M}, ɗ)$ be a metric $\mathbb{G}-$space and $\phi:\mathcal{M}\to\mathcal{M}$ be a continuous map. For $\delta>0$, a sequence $\{m_i, i\in\mathbb{Z}\}$ is named $(\mathbb{G},\delta)-$ periodic pseudo-trajectory of ϕ if $\forall i\geq 0$, $\exists g_i\in\mathbb{G}$ such that $ɗ(g_i\,\phi(m_i), m_{i+1})<\delta$, and for some integer $s>0$, we have $m_{rs+t}=m_t$, $r\geq 0$, $0\leq t<s$. A map ϕ has the $\mathbb{G}$Per.SP if $\forall\varepsilon>0$, $\exists\,\delta>0$ such that each $(\mathbb{G},\delta)-$periodic pseudo-trajectory of ϕ is able to be $(\mathbb{G},\varepsilon)-$shadowed with some point in $P_{\mathbb{G}}(\phi)$.

Next, we present notion of the $\mathbb{G}-$ average shadowing property ($\mathbb{G}$ ASP).

Definition 2.5

Let $(\mathcal{M}, ɗ)$ be a metric $\mathbb{G}-$space and $\phi:\mathcal{M}\to\mathcal{M}$ be a continuous map. For a real number δ, that is positive, a sequence of points $\{m_i: a<i<b\}$ in $\mathcal{M}$ is named $(\delta,\mathbb{G})-$average pseudo-trajectory for ϕ if $\forall\, i,\ a<i<b-1$, $\exists\, g_i\in\mathbb{G}$, and $\exists\, N\in\mathbb{Z},\ N>0$ and $N=N(\delta)$ such that $\forall\, n\geq N$ and $k\in\mathbb{N}$ we have

$$\frac{1}{n}\sum_{i=0}^{n-1} ɗ(g_i\,\phi(m_{i+k}), m_{i+k+1})<\delta,$$

The map ϕ has the $\mathbb{G}-$ average shadowing property ($\mathbb{G}$ ASP) if $\forall\,\varepsilon>0$, $\exists\delta>0$ such that each $(\delta,\mathbb{G})-$ average pseudo-trajectory $\{m_i: a<i<b\}$ is $\varepsilon-$shadowed in $\mathbb{G}-$average by a point m of $\mathcal{M}$, if $\forall\, i$, $\exists\, g_i\in\mathbb{G}$ such that

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} ɗ(\phi^{i}(m), g_i m_i)<\varepsilon.$$

Note that if ϕ is bijective we take $-\infty<a<b<\infty$, also, when ϕ is not bijective we take $0\leq a<b<\infty$.

Definition 2.6 [13]

Let $(\mathcal{M}, ɗ)$ be a metric $\mathbb{G}-$space. The map $\phi:\mathcal{M}\to\mathcal{M}$ is named $\mathbb{G}-$transitive if for any two of non-empty open subsets A and B of $\mathcal{M}$, $\exists\, i\in\mathbb{N}$ and $g\in\mathbb{G}$ such that the set $N_g(A\cap B)=\{i\in\mathbb{N}: g\,\phi^{i}(A)\cap B\neq\emptyset\}\neq\emptyset$. A homeomorphism ϕ is totally $\mathbb{G}-$transitive if ϕ^{i} is $\mathbb{G}-$transitive, $\forall i\geq 1$.

Definition 2.7 [14]

Let $(\mathcal{M}, ɗ)$ be a metric $\mathbb{G}-$space and $\phi:\mathcal{M}\to\mathcal{M}$ be a homeomorphism map then ϕ is said to be *topologically $\mathbb{G}-$mixing* if for any non-empty open subsets A and B of $\mathcal{M}$, $\exists\, k\in\mathbb{Z}$ such that $\forall\, n\geq k$, $\exists\, g_k\in\mathbb{G}$ satisfying $g_k\phi^{k}(A)\cap B\neq\emptyset$. We say that ϕ is $(\delta,\mathbb{G})-$*chain mixing* if there is an $N>0$ such that for any $m,y\in\mathcal{M}$ and any $n\geq N$, there is an $(\delta,\mathbb{G})-$pseudo-trajectory from m to y of length exactly n. If the map ϕ is $\delta-$chain mixing for every $\delta>0$, then ϕ is *chain mixing*, and the map ϕ is called *weakly $\mathbb{G}-$mixing* if the Cartesian product $\phi\times\phi$ is $\mathbb{G}\times\mathbb{G}-$transitive.

Main Results

Let $(\mathcal{M}, ɗ)$ be a metric $\mathbb{G}-$space and $\phi:\mathcal{M}\to\mathcal{M}$ be a continuous map. In this section, we prove some results for some chaotic properties of $\mathbb{G}-$periodic shadowing property ($\mathbb{G}$ Per.SP).

Proposition 3.1

If $\phi:\mathcal{M}\to\mathcal{M}$ has the $\mathbb{G}-$periodic shadowing property , then ϕ^{n} has the $\mathbb{G}-$periodic shadowing property , $\forall n\in\mathbb{N}$.

Proof :

Assume that ϕ has the $\mathbb{G}$ Per.SP, then $\forall\varepsilon>0$ is given there is a $\delta>0$. Let $\{m_i, i\in\mathbb{Z}\}$ be a $(\mathbb{G},\delta)-$periodic pseudo-trajectory of ϕ^{n}. We construct the sequence $\{\omega_i, i\in\mathbb{Z}\}$ with

$$\omega_t=\begin{cases} m_s & \text{for } t=ns,\\ \phi^{t-ns}(m_s) & \text{for } ns<t<n(s+1),\end{cases}$$

that means, $\{\omega_i, i\in\mathbb{Z}\}=\{\dots, m_0, \phi(m_0),\dots,\phi^{n-1}(m_0), m_1,\phi(m_1),\dots,\phi^{n-1}(m_1),\dots\}$is also a $\mathbb{G}-$periodic bisequence. It is simple to note that $\{\omega_i, i\in\mathbb{Z}\}$ is a $(\mathbb{G},\delta)-$pseudo-trajectory for ϕ. By the $\mathbb{G}$ Per.SP of ϕ find a $\mathbb{G}$-periodic point $p\in\mathcal{M}$ such that $\forall i\geq 0$, $\exists g_i\in\mathbb{G}$, then $ɗ(\phi^{i}(p), g_i\omega_i)<\varepsilon$. We pick $i=ns$ in the last inequality we obtain $\omega_i=m_s$, $g_i=g_s$ and $ɗ((\phi^{n})^{s}(p), g_s m_s)<\varepsilon$. Therefore ϕ^{n} has the $\mathbb{G}$ Per.SP.

Lemma 3.2

Let $\phi:\mathcal{M}\to\mathcal{M}$ be a uniform continuity homeomorphism map. Then ϕ has the $\mathbb{G}-$periodic shadowing property if and only if ϕ^{-1}has the $\mathbb{G}-$periodic shadowing property .

Proof:$(\Longrightarrow)$

Assume that ϕ has the $\mathbb{G}$ Per.SP. Let $\varepsilon>0$ be given and $\alpha>0$ be the number of the $\mathbb{G}$ Per.SP of ϕ. $\exists\,\alpha_1>0$ by the uniform continuity of ϕ such that: $ɗ(m,\omega)<\alpha_1$ implies that $ɗ(\phi(m),\phi(\omega))<\alpha$.

Let $\{\omega_i, i\in\mathbb{Z}\}$ be a $(\mathbb{G},\alpha_1)-$periodic pseudo-trajectory of ϕ^{-1}. Therefore, we have $\forall i\geq 0$, $\exists g_i\in\mathbb{G}$ such that $ɗ(g_i\,\omega_i,\phi(\omega_{i+1}))<\alpha$, and for some integer $s>0$, we have $m_{rs+t}=m_t$, $r\geq 0$, $0\leq t<s$.

We put $m_i=\omega_{-i}$, $\forall i\in\mathbb{Z}$. Clearly, $\{m_i, i\in\mathbb{Z}\}$ is $\mathbb{G}-$periodic such that $\forall i\geq 0$, $\exists g_i\in\mathbb{G}$ such that $ɗ(g_i\,\phi(m_i), m_{i+1})<\alpha$, and for some integer $s>0$, we have $m_{rs+t}=m_t$, $r\geq 0$, $0\leq t<s$. Therefore, by the $\mathbb{G}$ Per.SP for ϕ, there exists $\mathbb{G}-$periodic point ω such that $\forall i\in\mathbb{Z}$, $\exists\, g_i\in\mathbb{G}$, such that $ɗ(\phi^{i}(\omega), g_i m_i)<\varepsilon$.

Thus, $ɗ(\phi^{-i}(\omega), g_i m_{-i})<\varepsilon$, and this means that $ɗ((\phi^{-1})^{i}(\omega), g_i\omega_i)<\varepsilon$.

Hence ϕ^{-1} has the $\mathbb{G}$ Per.SP.

$(\Longleftarrow)$

Similarly ,we can easily show that if ϕ^{-1} has the $\mathbb{G}$ Per.SP then ϕ has the $\mathbb{G}$ Per.SP.

Proposition 3.3

Let $\phi:\mathcal{M}\to\mathcal{M}$ be a homeomorphism map. If ϕ has the $\mathbb{G}-$periodic shadowing property, then ϕ^{n} has the $\mathbb{G}-$periodic shadowing property , $\forall\, n\in\mathbb{Z}\setminus\{0\}$.

Proof:

We get this proof by using Proposition 3.1. and Lemma 3.2.

Definition 3.4 [8]

Let $(\mathcal{M}, ɗ)$ be a metric $\mathbb{G}-$space and $\phi:\mathcal{M}\longrightarrow\mathcal{M}$ be a homeomorphism map, then ϕ is called positively $\mathbb{G}-$expansive, if there exists real number $\rho>0$ such that $\forall\, m,y\in\mathcal{M}$ with $\mathbb{G}(m)\neq\mathbb{G}(y)$, there exists a integer number $k\geq 0$ such that $ɗ(\phi^{k}(u),\phi^{k}(v))>\rho$, $\forall\, u\in\mathbb{G}(m)$ and $v\in\mathbb{G}(y)$, ρ is then called a $\mathbb{G}-$expansive constant for ϕ.

Theorem 3.5

Let $\phi:\mathcal{M}\to\mathcal{M}$ be an $\mathbb{G}-$expansive. If ϕ has the $\mathbb{G}-$ shadowing property , then ϕ has the $\mathbb{G}-$periodic shadowing property.

Proof :

Suppose $\rho>0$ be a $\mathbb{G}$ –expansive constant for ϕ. It is enough to prove it for $\varepsilon\leq\frac{\rho}{2}$. Let $0<\varepsilon<\frac{\rho}{2}$ is given, since ϕ has the $\mathbb{G}$SP then $\exists\,\delta>0$ such that if $\{m_i, i\in\mathbb{Z}\}$ is a $(\mathbb{G},\delta)-$periodic pseudo-trajectory of ϕ, then $\exists\,\omega$ such that $\forall i\in\mathbb{Z}$, $\exists g_i\in\mathbb{G}$, such that $ɗ(\phi^{i}(\omega), g_i m_i)<\varepsilon$ … … … … … (1).

Assume that $\{m_i, i\in\mathbb{Z}\}$ is a $\alpha-$periodic sequence, that is , $m_{i+\alpha}=m_i$, $\forall i\in\mathbb{Z}$ and $g_i\in\mathbb{G}$ then $ɗ(\phi^{i+\alpha}(\omega), g_i m_i)<\varepsilon$.

Put $z=\phi^{\alpha}(\omega)$ then, we have $ɗ(\phi^{i}(z), g_i m_i)<\varepsilon$, $\forall i\in\mathbb{Z}$ and $g_i\in\mathbb{G}$ … … … … (2).

From (1) and (2), we get

$ɗ(\phi^{i}(z),\phi^{i}(\omega))<ɗ(\phi^{i}(\omega), g_i m_i)+ɗ(\phi^{i}(z), g_i m_i)<\varepsilon+\varepsilon<2\varepsilon\leq\rho$, $\forall i\in\mathbb{Z}$.

Therefore, the $\mathbb{G}-$expansivity of ϕ produces $z=\omega=\phi^{\alpha}(\omega)$.

Thus ϕ has the $\mathbb{G}$ Per.SP.

Definitions 3.6 [7]

Let$(\mathcal{M}, ɗ)$ be a metric $\mathbb{G}-$space and $\phi:\mathcal{M}\to\mathcal{M}$ be a continuous map. For a positive real number δ, a sequence of points $\{m_i: a<i<b\}$ in $\mathcal{M}$ is said to be $(\delta,\mathbb{G})-$pseudo-trajectory for ϕ, if $\forall\, i,\ a<i<b-1$, $\exists\, g_i\in\mathbb{G}$ such that $ɗ(g_i\,\phi(m_i), m_{i+1})<\delta$.

For a given $\varepsilon>0$, a $(\delta,\mathbb{G})-$pseudo trajectory $\{m_i: a<i<b\}$ for ϕ is said to $\varepsilon-$traced by a point m of $\mathcal{M}$ if $\forall i, a<i<b$, $\exists\, p_i\in\mathbb{G}$ such that $ɗ(\phi^{i}(m), p_i m_i)<\varepsilon$, the map ϕ has the $\mathbb{G}-$*pseudo trajectory tracing property* **(** $\mathbb{G}$PTTP **)** if $\forall\,\varepsilon>0$, $\exists\,\delta>0$ such that every $(\delta,\mathbb{G})-$pseudo -trajectory for ϕ is $\varepsilon-$ traced by a point $m\in\mathcal{M}$. Note that if ϕ is bijective we take $-\infty<a<b<\infty$,also, when ϕ is not bijective we take $0\leq a<b<\infty$.

Definitions 3.7
Let $(\mathcal{M},ɗ)$ be a metric $\mathbb{G}-$space and $\phi:\mathcal{M}\to\mathcal{M}$ be a continuous map. For a positive real number δ, a sequence of points $\{m_i:\ 0\le i\le n\}$ in $\mathcal{M}$ is said to be finite $(\delta,\mathbb{G})-$pseudo-trajectory for ϕ if $\forall\, i,\ 0\le i<n-1,\ \exists\, g_i\in\mathbb{G}$ such that $ɗ(g_i\,\phi\,(m_i),\ m_{i+1})<\delta$.
For a given $\varepsilon>0$ and $\delta>0$, a finite $(\delta,\mathbb{G})-$pseudo-trajectory $\{m_i:\ 0\le i\le n\}$ for ϕ is said to $\varepsilon-$shadowed by a point $m\in\mathcal{M}$ if $\forall\, i,\ 0\le i<n-1,\ \exists\, p_i\in\mathbb{G}$ such that $ɗ(\phi^i(m),\ p_i m_i)<\varepsilon$, the map ϕ has the **finite** $\mathbb{G}$SP, if $\forall\,\varepsilon>0,\ \exists\,\delta>0$ such that each finite $(\delta,\mathbb{G})-$pseudo-trajectory for ϕ is $\varepsilon-$shadowed by a point $m\in\mathcal{M}$.
Lemma 3.8
Let $(\mathcal{M},ɗ)$ be a compact metric $\mathbb{G}-$space, $\phi:\mathcal{M}\to\mathcal{M}$ be a continuous map and ϕ has the finite $\mathbb{G}-$ shadowing property on $Y\subset\mathcal{M}$, then ϕ has the $\mathbb{G}-$pseudo-trajectory tracing properly($\mathbb{G}$PTTP) on Y.
Proof:
Let $\varepsilon>0$ be given, since ϕ has the finite $\mathbb{G}$ SP on $Y\subset\mathcal{M}$ then $\exists\,\delta>0$, such that $S=\{m_i:\ i\in\mathbb{Z}\}\subset Y$ be a $(\delta,\mathbb{G})-$pseudo-trajectory for ϕ. Fix $t>0$ and $m_i'=m_{i-t}$. By our hypothesis, there is a point $y_t\in\mathcal{M}$ and $\exists\, g_i\in\mathbb{G}$ such that $ɗ(\phi^i(y_t),g_i m_i')<\varepsilon,\ 0\le i\le 2t$. Let $\omega_t=\phi^t(y_t)$, then $ɗ(\phi^i(\omega_t),g_i m_i)<\varepsilon,\ |i|\le t$. Let ω be a limit point of the sequence ω_t. Passing to the limit as $m\to\infty$ in $ɗ(\phi^i(\omega_t),g_i m_i)<\varepsilon$, we see that $ɗ(\phi^i(\omega),p_i m_i)\le\varepsilon$, $i\in\mathbb{Z}$ and $p_i\in\mathbb{G}$.
Thus S is $2\varepsilon-$shadowed by ω. Hence ϕ has the $\mathbb{G}$PTTP on Y.

Theorem 3.9
Let $(\mathcal{M},ɗ)$be a compact metric $\mathbb{G}-$space and $\phi:\mathcal{M}\to\mathcal{M}$ be a $\mathbb{G}-$chain transitive map. If ϕ has the $\mathbb{G}-$periodic shadowing property, then ϕ has the $\mathbb{G}-$shadowing property and $\mathbb{G}-$transitivity.
Proof :
By Lemma 3.8, it is enough to prove that ϕ has the finite $\mathbb{G}$SP. Let $\varepsilon>0$ be given, since ϕ has the $\mathbb{G}$ Per.SP then $\exists\,\delta>0$, such that $\{m_i:\ 0\le i\le t\}\subset\mathcal{M}$ is a finite a $(\delta,\mathbb{G})-$pseudo-trajectory for ϕ, then by $\mathbb{G}-$chain transitivity of ϕ there is a $\delta-$chain from m_t to m_0, of length n, that is, there are points $x_0=m_t,x_1,\dots,x_n=m_0$, so that $ɗ(g_i\,\phi\,(x_i),\ x_{i+1})<\delta$,
for $0\le i\le n-1$. Add the points x_i, $1\le i\le n-1$ to m_i's to get a $(t+n)\,(\delta,\mathbb{G})-$pseudo-trajectory
$\{m_0,m_1,\dots,m_t,x_1,x_2,\dots,x_{n-1}\}$.
Now we expand the $(t+n)\,(\delta,\mathbb{G})-$pseudo-trajectory to a periodic sequence $S=\{y_i:\ i\in\mathcal{N}_0\}$. Thus S is a$(t+n)\,(\delta,\mathbb{G})-$periodic pseudo-trajectory, hence the $\mathbb{G}$ Per.SP leads that is a $\mathbb{G}-$periodic point $\omega\in\mathcal{M}$ such that $ɗ(\phi^i(\omega),g_i y_i)<\varepsilon$, $i\in\mathcal{N}_0$ and $g_i\in\mathbb{G}$. For $0\le i\le t$, we gather $ɗ(\phi^i(\omega),x_i m_i)<\varepsilon$. Hence ϕ has the $\mathbb{G}$SP and so $\mathbb{G}-$transitivity.

Remark 3.10
We note that Theorem 3.9 is also held when ϕ is $\mathbb{G}-$chain mixing. further, so as there are subshifts of finite kind without being $\mathbb{G}-$chain transitive then the $\mathbb{G}-$chain transitivity precondition in Theorem 3.9 is not essential precondition.

Corollary 3.11
Let$(\mathcal{M},ɗ)$ be a compact metric $\mathbb{G}-$space, $\phi:\mathcal{M}\to\mathcal{M}$ an $\mathbb{G}-$expansive homeomorphism map and $\mathbb{G}-$chain transitive . Then ϕ has the $\mathbb{G}-$shadowing property iff ϕ has the $\mathbb{G}-$periodic shadowing property.
Proof :
We get the proof by Theorem 3.5 and Theorem 3.9.

For proof the converse of Proposition 3.1, we must proof if ϕ is $\mathbb{G}-$chain mixing, then ϕ^n, for each $n\in\mathbb{N}$, is $\mathbb{G}-$chain transitive, that is , ϕ is totally $\mathbb{G}-$chain transitive.
Proposition 3.12
Let$(\mathcal{M},ɗ)$ be a compact metric $\mathbb{G}-$space and $\phi:\mathcal{M}\to\mathcal{M}$ be a uniformly continuous map. If ϕ is $\mathbb{G}-$chain mixing, then ϕ^n is $\mathbb{G}-$chain transitive for each $n\in\mathbb{N}$.
Proof :
Assume that ϕ is $\mathbb{G}-$chain mixing. Let $n>0$ and pick $\varepsilon>0$, by uniform continuity then $\exists\,\alpha>0$, and $p_1,p_2\in\mathcal{M}$ such that if $ɗ(p_1,p_2)<\alpha$ implies $ɗ\left(\phi^i(p_1),\phi^i(p_2)\right)<\frac{\varepsilon}{n}$, for $i=1,\dots,n$.
Since ϕ is $\mathbb{G}-$chain mixing, we pick a $\alpha-$chain from m to y of length of $D=Ln$, $m=\ell_0,\ell_1,\dots,\ell_D=y$, $ɗ(g_i\phi(\ell_i),\ell_{i+1})<\alpha$, for $i=0,1,\dots,D-1$ and $g_i\in\mathbb{G}$.
So, by triangle inequality, we prove that the next sequence is a $\varepsilon-$chain for ϕ^n from m to y, $m=\ell_0,\ell_n,\ell_{2n},\dots,\ell_{D-n},\ \ell_D=y$.
Then ϕ^n is $\mathbb{G}-$chain transitive for each $n\in\mathbb{N}$.

Now we prove the converse of Proposition 3.1.
Proposition 3.13
Let$(\mathcal{M},ɗ)$ be a compact metric $\mathbb{G}-$space, $\phi:\mathcal{M}\to\mathcal{M}$ is an $\mathbb{G}-$expansive map and $\mathbb{G}-$chain mixing. If ϕ^n has the $\mathbb{G}-$periodic shadowing property for some $n\in\mathbb{N}$, such that $n\ne 1$ then ϕ has the $\mathbb{G}-$periodic shadowing property.
Proof :
The map ϕ^n is $\mathbb{G}-$chain transitive by Proposition 3.12 and it has the $\mathbb{G}$ Per.SP by Theorem 3.9 and so ϕ has the $\mathbb{G}$ SP. Since ϕ is an $\mathbb{G}-$expansive map then by Theorem 3.5, ϕ has the $\mathbb{G}$ Per.SP.

Remark 3.14
If we change the $\mathbb{G}-$chain mixing by the $\mathbb{G}-$chain transitivity of ϕ^n, the Proposition 3.13, is held under feebler suppositions.
Theorem 3.15 [15]
Let$(\mathcal{M},ɗ)$ be a compact metric $\mathbb{G}-$space and $\phi:\mathcal{M}\longrightarrow\mathcal{M}$ be pseudo-equivariant with dense set of $\mathbb{G}_\phi-$periodic points on a compact metric $\mathbb{G}-$ space $(\mathcal{M},ɗ)$. If ϕ has the $\mathbb{G}-$ average shadowing property, then ϕ is weakly $\mathbb{G}-$mixing.
Corollary 3.16
Let$(\mathcal{M},ɗ)$ be a metric $\mathbb{G}-$space and $\phi:\mathcal{M}\longrightarrow\mathcal{M}$ be pseudo-equivariant surjective map on a compact metric $\mathbb{G}-$ space $(\mathcal{M},ɗ)$ and with dense set of $\mathbb{G}_\phi-$periodic points which has the $\mathbb{G}-$periodic shadowing property. If ϕ has the $\mathbb{G}-$ average shadowing property, then ϕ is weakly $\mathbb{G}-$ mixing.
Proof:
By Theorem 3.9, we get, if ϕ has the $\mathbb{G}$ Per.SP, then ϕ has the $\mathbb{G}$ SP. By Theorem 3.15, if ϕ has the $\mathbb{G}$ ASP, then ϕ is weakly $\mathbb{G}-$mixing.

Definitions 3.17 [16]
Let a group $\mathbb{G}$ act on a Hausdorff topological space $\mathcal{M}$. The points $m,y\in\mathcal{M}$ are named proximal with reference to the action of $\mathbb{G}$ (or $\mathbb{G}-$proximal) if there is a sequence $\{g_k\}$ in $\mathbb{G}$ such that the sequences $\{g_k m\}$ and $\{g_k y\}$ converge to the same point. If m and y are not $\mathbb{G}-$proximal, they are $\mathbb{G}-$distal. An action of a group $\mathbb{G}$ on a space $\mathcal{M}$ is said to be proximal if any two points in $\mathcal{M}$ are $\mathbb{G}-$proximal, and it is distal if any two distinct points in $\mathcal{M}$ are $\mathbb{G}-$distal.
Theorem 3.18 [17]
Let $(Y,ɗ)$ be a metric space and $\phi:Y\to Y$ be a continuous map. if ϕ is distal, then each $y\in Y$ is an almost periodic point.

Theorem 3.19
Let$(\mathcal{M},ɗ)$ be a metric $\mathbb{G}-$ space and $\phi:\mathcal{M}\to\mathcal{M}$ be a continuous map. if ϕ is $\mathbb{G}-$ distal, then each $m\in\mathcal{M}$ is an almost $\mathbb{G}-$ periodic point.
Proof:
This proof is similar the technique of proof as Theorem 3.18.

Lemma 3.20
Let $(\mathcal{M},ɗ)$ be a compact metric $\mathbb{G}-$space has at least two points and $\phi:\mathcal{M}\longrightarrow\mathcal{M}$ be a $\mathbb{G}-$distal homeomorphism. If ϕ^n is $\mathbb{G}-$chain transitive for each positive integer n, then ϕ does not have the $\mathbb{G}-$ shadowing property.
Proof :
Suppose on the opposite that ϕ has the $\mathbb{G}-$shadowing property. Let m and y be two distinct points in $\mathcal{M}$. Pick $\varepsilon>0$ such that $B(m,\varepsilon)\cap B(y,\varepsilon)=\emptyset$, where $B(m,\varepsilon)=\{s\in\mathcal{M}:\ ɗ(s,m)<\varepsilon\}$. Let $\gamma\in(0,\varepsilon)$ as a number in the definition of the $\mathbb{G}$SP for ϕ . Since ϕ is a $\mathbb{G}-$distal homeomorphism, by Theorem 3.18 every point of $\mathcal{M}$ is an almost $\mathbb{G}-$periodic point of ϕ . Hence $\exists\, t\in\mathbb{N}$ such that $\phi^t(m)\in B(m,\gamma)$.

Let $\omega_{nt+j} = \phi^{j}(m)$ for $0 \leq j < t$ and $n \geq 0$. We easy note that the sequence $\{\omega_i, 0 \leq i < \infty\}$is a $(\mathbb{G},\gamma)-$periodic pseudo-trajectory of ϕ. Since ϕ^t is $\mathbb{G}-$chain transitive, there is a $\gamma-$chain of $\phi^t$from y to m, say $\{y = y_0, y_1, \ldots, y_\ell = m\}$. Thus,$\forall\, g_i \in \mathbb{G}$, then $\{g_0y_0, \phi\,(g_0y_0), \ldots, \phi^{t-1}(g_0y_0), g_1y_1, \phi\,(g_1y_1), \ldots, \phi^{t-1}(g_1y_1), \ldots, g_{\ell-1}\, y_{\ell-1}, \phi\,(g_{\ell-1}y_{\ell-1}),$ $\ldots, \phi^{t-1}(g_{\ell-1}y_{\ell-1}), g_0\, \omega_0, g_2\omega_2, \ldots, g_i\omega_i, \ldots\}$ is a $(\mathbb{G},\gamma)-$pseudo-trajectory of ϕ. Hence, it can be $\varepsilon-$shadowed by some point z in $\mathcal{M}$ and we have$\mathrm{d}(gz, gy) < \varepsilon$ and $\mathrm{d}\big(g\phi^{(\ell+k)t}(z), gm\big) < \varepsilon$, $\forall k \geq 0$ and $g \in \mathbb{G}$ ……… (3)

It is easy going to check up that ϕ^t is a $\mathbb{G}-$distal homeomorphism as ϕ is. The point z is also an almost $\mathbb{G}-$periodic point of ϕ^t. But, it is produced from (3) that $z \in B(y,\varepsilon)$ and $g\phi^{(\ell+k)t}(z) \in B(m,\varepsilon)$ $\forall k \geq 0$. So it is a contradicted .

Hence ϕ never has the $\mathbb{G}$SP.

Corollary 3.21

Let $(\mathcal{M}, \mathrm{d})$ be a compact metric $\mathbb{G}-$space has at least two points and $\phi: \mathcal{M} \longrightarrow \mathcal{M}$ be a $\mathbb{G}-$distal homeomorphism. If ϕ is $\mathbb{G}-$chain mixing, then ϕ never has the $\mathbb{G}-$periodic shadowing property.

Proof:

Assume on the opposite that ϕ has the $\mathbb{G}$ Per.SP. So as ϕ is $\mathbb{G}-$chain mixing, by Proposition 3.12 leads to ϕ^n is $\mathbb{G}-$chain transitive $\forall\, n \in \mathbb{N}$, and by Theorem 3.9 then ϕ has the $\mathbb{G}$ SP. But, this contradicts with Lemma 3.20.

Hence ϕ does not have the $\mathbb{G}$ Per.SP.

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