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Review Article

A TRIPLE FIXED POINT THEOREM OF CARISTI TYPE CONTRACTION FOR MULTI VALUED MAPS IN A HAUSSDORFF METRIC SPACE

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Abstract
The main aim of this paper is to obtain a unique common tripled fixed point of caristi type caristi type ontraction for multi valued mappings in a Haussdorff metric space

Keywords: Metric space , compatible maps, tripled fixed point, Haussdorff metric.

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INTRODUCTION

The concept of standard metric space is a fundamental tool in topology,functional analysis and nonlinear analysis. This structure has attracted a considerable attention from mathematicians because of the development of the fixed point theory in standard metric space. Since Banach Introduced this theory in 1922([10]), it has been extended and generalized by several authors. Caristi type fixed point theorem is one of these generalizations. It is a modified ϵ -variation principle of Ekeland([9]). In 1976, Caristi proved the following famous fixed point theorem.

Theorem 1.1 *[6] Let (X,d) be complete metric space and $f: X \to R$ be lower semi continuous function and bounded below function. A mapping $T: X \to X$ is said to be Caristi type map on X dominated by f if T satisfies $d(x,Tx) \le f(x) - f(Tx)$ for each $x \in X$. Then T has a fixed point.*

S.B.Nadler introduced the concept of multivalued contraction mappings in the year 1969([11]).

Definition 1.2 *([11]) Let (X,d) be a metric space. We define the Hausdorff metric on CB(X) induced by d. That is $H(A,B) = max\{\sup_{l\in A} d(l,B), \sup_{m\in B} d(m,A)\}$ for all $A,B \in CB(X)$, where $CB(X)$ denotes the family of all nonempty closed and bounded subsets of X and $d(l,B) = inf\{d(l,b): b \in B\}$, for all $l \in X$.*

Definition 1.3 *([11]) Let (X,d) be a metric space. A map $T: X \to CB(X)$ is said to be multivalued contraction if there exists $0 \le \alpha < 1$ such that $H(Tl,Tm) \le \alpha d(l,m)$, for all $l,m \in X$.*

Lemma 1.4 *([8]) Let X be a nonempty set and $g: X \to X$ be a mapping. then there exists a subset $E \subseteq X$ such that $g(E) = g(X)$ and $g: E \to E$ is one one.*

Now we give the following definitions for hybrid pair of mappings.

Definition 1.5 *([7])Let X be a non empty set, $T: X \times X \times X \to 2^X$ (collection of non empty substes of X) and $f: X \to X$.*

(i) The point $(l,m,n) \in X \times X \times X$ is called a tripled fixed point of T if

$$l \in T(l,m,n)$$
$$m \in T(m,l,m)$$
$$n \in T(n,m,l)$$

(ii) The point $(l,m,n) \in X \times X \times X$ is called a tripled coincident point of T and f if

$$fl \in T(l,m,n)$$
$$fm \in T(m,l,m)$$
$$fn \in T(n,m,l)$$

(iii) The point $(l,m,n) \in X \times X \times X$ is called a tripled common fixed point of T and f if

$$l = fl \in T(l,m,n)$$
$$m = fm \in T(m,l,m)$$
$$n = fn \in T(n,m,l)$$

Definition 1.6 *[7] Let $T: X \times X \times X \to X$ be a multi valued map and f be self map on X. The hybrid pair $\{T,f\}$ is called $w-$compatible if $f(T(l,m,n)) \subseteq T(fl,fm,fn)$ whenever (l,m,n) is tripled coincidence point of T and f.*

Lemma 1.7 *(See [5]) Let $\lhd$ be a reflexive relation on a nonempty set M and $\phi: M \to R$ a function bounded from below, then $\chi \lhd \gamma$ and $\chi \neq \gamma$; then $\phi(\chi) > \phi(\gamma)$.*

Throughout this paper, we assume that $\zeta: [0,\infty) \to [0,\infty)$ is an upper semi continuous function.

Now we prove our main results.

RESULTS AND DISCUSSIONS

Theorem 2.1 *Let (X,d) be a complete metric space and let $S: X \times X \times X \to CB(X)$ be a set valued mapping satisfies*

$$H\big(S(l,m,n), S(a_1,b_1,c_1)\big) \le \max\left\{\begin{array}{l}\zeta\left(\max\big(\varsigma(l,a_1),\varsigma(m,b_1),\varsigma(n,c_1)\big)\right),\\ \zeta\left(\max\big(\varsigma(\alpha_1,\beta_1),\varsigma(\alpha_2,\beta_2),\varsigma(\alpha_3,\beta_3)\big)\right)\end{array}\right\}$$

$$\begin{bmatrix}\max(\varsigma(l,a_1),\varsigma(m,b_1),\varsigma(n,c_1)))\\ -\max(\varsigma(\alpha_1,\beta_1),\varsigma(\alpha_2,\beta_2),\varsigma(\alpha_3,\beta_3))\end{bmatrix}$$

for some $\alpha_1 \in S(l,m,n), \alpha_2 \in S(m,l,m), \alpha_3 \in S(n,m,l)$ and $\beta_1 \in S(a_1,b_1,c_1), \beta_2 \in S(b_1,a_1,c_1), \beta_3 \in S(c_1,b_1,a_1)$. Where $\varsigma: X \times X \to [0,\infty)$ is lower semi continuous function and $\zeta: [0,\infty) \to [0,\infty)$ be an upper semi continuous function. Then S has a tripled fixed point.

Proof: Define a relation $\lhd$ on X as follows:

$$S(l,m,n) \lhd S(a_1,b_1,c_1)$$

$$\Leftrightarrow H\big(S(l,m,n), S(a_1,b_1,c_1)\big) \le \max\left\{\begin{matrix}\zeta\left(\max\big(\varsigma(l,a_1),\varsigma(m,b_1),\varsigma(n,c_1)\big)\right),\\ \zeta\left(\max\big(\varsigma(\alpha_1,\beta_1),\varsigma(\alpha_2,\beta_2),\varsigma(\alpha_3,\beta_3)\big)\right)\end{matrix}\right\}$$

$$\begin{bmatrix}\max(\varsigma(l,a_1),\varsigma(m,b_1),\varsigma(n,c_1)))\\ -\max(\varsigma(\alpha_1,\beta_1),\varsigma(\alpha_2,\beta_2),\varsigma(\alpha_3,\beta_3))\end{bmatrix}$$

Then clearly $\lhd$ is a reflexive relation on X.

Let $l_0, m_0, n_0 \in X$ be arbitrary points in X.

Choose $l_1 \in S(l_0,m_0,n_0), m_1 \in S(m_0,l_0,m_0), n_1 \in S(n_0,m_0,l_0)$

Since S is compact valued maps so there exists $l_2 \in S(l_1,m_1,n_1)$,$m_2 \in S(m_1,l_1,m_1)$ and $n_2 \in S(n_1,m_1,l_1)$ such that

$$d(l_1,l_2) \le H\big(S(l_0,m_0,n_0), S(l_1,m_1,n_1)\big) \le \max\left\{\begin{matrix}\zeta\left(\max\big(\varsigma(l_0,l_1),\varsigma(m_0,m_1),\varsigma(n_0,n_1)\big)\right),\\ \zeta\left(\max\big(\varsigma(l_1,l_2),\varsigma(m_1,m_2),\varsigma(n_1,n_2)\big)\right)\end{matrix}\right\}$$

$$\begin{bmatrix}\max(\varsigma(l_0,l_1),\varsigma(m_0,m_1),\varsigma(n_0,n_1)))\\ -\max(\varsigma(l_1,l_2),\varsigma(m_1,m_2),\varsigma(n_1,n_2)))\end{bmatrix}$$

and

$$d(m_1,m_2) \le H\big(S(m_0,l_0,m_0), S(m_1,l_1,m_1)\big) \le \max\left\{\begin{matrix}\zeta\left(\max\big(\varsigma(m_0,m_1),\varsigma(l_0,l_1),\varsigma(m_0,m_1)\big)\right),\\ \zeta\left(\max\big(\varsigma(l_1,l_2),\varsigma(m_1,m_2),\varsigma(n_0,n_1)\big)\right)\end{matrix}\right\}$$

$$\begin{bmatrix}\max(\varsigma(m_0,m_1),\varsigma(l_0,l_1),\varsigma(m_0,m_1)))\\ -\max(\varsigma(l_1,l_2),\varsigma(m_1,m_2),\varsigma(n_0,n_1)))\end{bmatrix}$$

Also

$$d(n_1,n_2) \le H\big(S(n_0,m_0,l_0), S(n_1,m_1,l_1)\big) \le \max\left\{\begin{matrix}\zeta\left(\max\big(\varsigma(l_0,l_1),\varsigma(m_0,m_1),\varsigma(n_0,n_1)\big)\right),\\ \zeta\left(\max\big(\varsigma(l_1,l_2),\varsigma(m_1,m_2),\varsigma(n_0,n_1)\big)\right)\end{matrix}\right\}$$

$$\begin{bmatrix}\max(\varsigma(l_0,l_1),\varsigma(m_0,m_1),\varsigma(n_0,n_1)))\\ -\max(\varsigma(l_1,l_2),\varsigma(m_1,m_2),\varsigma(n_0,n_1)))\end{bmatrix}.$$

Therefore

$$\max\{d(l_1,l_2), d(m_1,m_2), d(n_1,n_2)\} \le \max\left\{\begin{matrix}\zeta\left(\max\big(\varsigma(l_0,l_1),\varsigma(m_0,m_1),\varsigma(n_0,n_1)\big)\right),\\ \zeta\left(\max\big(\varsigma(l_1,l_2),\varsigma(m_1,m_2),\varsigma(n_0,n_1)\big)\right)\end{matrix}\right\}$$

$$\begin{bmatrix}\max(\varsigma(l_0,l_1),\varsigma(m_0,m_1),\varsigma(n_0,n_1)))\\ -\max(\varsigma(l_1,l_2),\varsigma(m_1,m_2),\varsigma(n_0,n_1)))\end{bmatrix}.$$

Continuing in this way we can obtain sequences $\{l_k\},\{m_k\},\{n_k\}$ in X such that $l_{k+1} \in S(l_k,m_k,n_k), m_{k+1} \in S(m_k,l_k,m_k)$ and $n_{k+1} \in S(n_k,m_k,l_k)$ such that

$$\max\{d(l_k,l_{k+1}), d(m_k,m_{k+1}), d(n_k,n_{k+1})\} \le \max\left\{\begin{matrix}\zeta\left(\max\big(\varsigma(l_{k-1},l_k),\varsigma(m_{k-1},m_k),\varsigma(n_{k-1},n_k)\big)\right),\\ \zeta\left(\max\big(\varsigma(l_k,l_{k+1}),\varsigma(m_k,m_{k+1}),\varsigma(n_k,n_{k+1})\big)\right)\end{matrix}\right\}$$

$$\begin{bmatrix}\max(\varsigma(l_{k-1},l_k),\varsigma(m_{k-1},m_k),\varsigma(n_{k-1},n_k)))\\ -\max(\varsigma(l_k,l_{k+1}),\varsigma(m_k,m_{k+1}),\varsigma(n_k,n_{k+1})))\end{bmatrix}.$$

Since $l_k \ne l_{k+1}$, $m_k \ne m_{k+1}$, $n_k \ne n_{k+1}$ so from Lemma 1.7 we have $\{\varsigma(l_k,l_{k+1})\}$ and $\{\varsigma(m_k,m_{k+1})\}$,$\{\varsigma(n_k,n_{k+1})\}$, are non increasing.

Let

$\text{Lim}_{k\to\infty}\{\varsigma(l_k,l_{k+1})\} = \lambda_1, \lim_{k\to\infty}\{\varsigma(m_k,m_{k+1})\} = \lambda_2$,

$\lim_{k\to\infty}\{\varsigma(n_k,n_{k+1})\} = \lambda_3$, for some $\lambda_1,\lambda_2,\lambda_3 \ge 0$.

If $\lambda_1,\lambda_2,\lambda_3 = 0$ then we get a contradiction. So $\lambda_1,\lambda_2,\lambda_3 > 0$.

Since ζ is upper semi continuous function so we have

$\lim_{k\to\infty}\sup\zeta(\{\varsigma(l_k,l_{k+1})\}) = \zeta(\lambda_1)$

,$\lim_{k\to\infty}\sup\zeta(\{\varsigma(m_k,m_{k+1})\}) = \zeta(\lambda_2)$,

$\lim_{k\to\infty}\sup\zeta(\{\varsigma(n_k,n_{k+1})\}) = \zeta(\lambda_3)$.

So for any $q \in N$ with $k \ge k_0$ we have $\lim_{k\to\infty}\sup\zeta(\{\varsigma(l_k,l_{k+1})\}) = \zeta(\lambda_1) + 1$, $\lim_{k\to\infty}\sup\zeta(\{\varsigma(m_k,m_{k+1})\}) = \zeta(\lambda_2) + 1$, $\lim_{k\to\infty}\sup\zeta(\{\varsigma(n_k,n_{k+1})\}) = \zeta(\lambda_3) + 1$.

Therefore

$$\max\{d(l_k,l_{k+1}), d(m_k,m_{k+1}), d(n_k,n_{k+1})\} \le \max[\zeta(\lambda_1)+1, \zeta(\lambda_2)+1, \zeta(\lambda_3)+1]$$

$$\begin{bmatrix}\max(\varsigma(l_{k-1},l_k),\varsigma(m_{k-1},m_k),\varsigma(n_{k-1},n_k)))\\ -\max(\varsigma(l_k,l_{k+1}),\varsigma(m_k,m_{k+1}),\varsigma(n_k,n_{k+1})))\end{bmatrix}.$$

As $k \to \infty$, $\max\{d(l_k,l_{k+1}), d(m_k,m_{k+1}), d(n_k,n_{k+1})\} \to 0$.

Now for $q > k$, and as $q,k \to \infty$ $\max\{d(l_k,l_k), d(m_k,m_k), d(n_k,n_q)\}$

$$= \max\{d(l_k,l_{k+1}), d(m_k,m_{k+1}), d(n_k,n_{k+1})\} + \max\{d(l_{k+1},l_{k+2}), d(m_{k+1},m_{k+2}), d(n_{k+1},n_{n+2})\} + \cdots + \max\{d(l_{q-1},l_q), d(m_{q-1},m_q), d(n_{k-1},n_k)\}. = 0.$$

This shows $\{l_k\},\{m_k\},\{n_k\}$ are Cauchy sequence in X.

Since X is complete, there exists $l,m,n \in X$ such that

$\lim_{k\to\infty}\{l_k\} \to l, \lim_{k\to\infty}\{m_k\} \to m$ and $\lim_{k\to\infty}\{n_k\} \to n$.

Now consider

$$H\big(S(l_k,m_k,n_k), S(l,m,n)\big) \le \max\left\{\begin{matrix}\zeta\left(\max\big(\varsigma(l_k,l),\varsigma(m_k,m),\varsigma(n_k,n)\big)\right),\\ \zeta\left(\max\big(\varsigma(\alpha_1,\beta_1),\varsigma(\alpha_2,\beta_2),\varsigma(\alpha_3,\beta_3)\big)\right)\end{matrix}\right\}$$

$$\begin{bmatrix}\max(\varsigma(l_k,l),\varsigma(m_k,m),\varsigma(n_k,n)))\\ -\max\big(\varsigma(\alpha_1,\beta_1),\varsigma(\alpha_2,\beta_2),\varsigma(\alpha_3,\beta_3)\big)\end{bmatrix}$$

$$\leq \max\left\{\begin{matrix}\zeta\left(\max(\varsigma(l_k,l),\varsigma(m_k,m),\varsigma(n_k,n))\right), \\ \zeta\left(\max(\varsigma(\alpha_1,\beta_1),\varsigma(\alpha_2,\beta_2),\varsigma(\alpha_3,\beta_3))\right)\end{matrix}\right\}$$

$$[\max(\varsigma(l_k,l),\varsigma(m_k,m),\varsigma(n_k,n)))].$$

Letting $k \to \infty$ we have

$H(S(l_k,m_k,n_k),S(l,m,n))$

$$\leq \max\left\{\begin{matrix}\zeta\left(\max(\varsigma(l,l),\varsigma(m,m),\varsigma(n,n))\right), \\ \zeta\left(\max(\varsigma(\alpha_1,\beta_1),\varsigma(\alpha_2,\beta_2),\varsigma(\alpha_3,\beta_3))\right)\end{matrix}\right\}$$

$$[\max(\varsigma(l,l),\varsigma(m,m),\varsigma(n,n)))]$$

$= 0.$

Therefore $H(S(l_k,m_k,n_k),S(l,m,n)) = 0$.

Similarly we can prove that

$H(S(m_k,l_k,m_k),S(m,l,m)) = 0$ and

$H(S(n_k,m_k,l_k),S(n,m,l)) = 0$.

Since $l_{k+1} \in S(l_k,m_k,n_k), m_{k+1} \in S(m_k,l_k,m_k)$

and $n_{k+1} \in S(n_k,m_k,l_k)$,

so as $k \to \infty$

we have

$d(l_{k+1},S(l,m,n)) = \inf\{d(l_{k+1},a): a \in S(l,m,n)\}$,

$d(m_{k+1},S(m,l,m)) = \inf\{d(m_{k+1},b): b \in S(m,l,m)\}$ and

$d(n_{k+1},S(n,m,l)) = \inf\{d(n_{k+1},c): c \in S(n,m,l)$.

Hence there exists sequences $p_k \in S(l,m,n), w_k \in S(m,l,m)$ and $r_k \in S(n,m,l)$ such that $\lim_{k\to\infty} d(l_{k+1},p_n) = 0$,

$\lim_{k\to\infty} d(m_{k+1},w_k) = 0$ and $\lim_{k\to\infty} d(n_{k+1},r_n) = 0$.

It remains to prove that as $k \to \infty, p_k \to l, w_k \to m, r_k \to n$.

Suppose that p_k does not converges to l. Now as $k \to \infty$

$$d(p_k,l) < d(p_k,l_{k+1}) + d(l_{k+1},l)$$

$$< d(p_k,l) + d(l,l)$$

$$< d(p_k,l).$$

Therefore $d(p_k,l) < d(p_k,l)$, which is a contradiction. Hence $\lim_{k\to\infty} p_k = l$.

Similarly we can prove that $\lim_{k\to\infty} w_k = m, \lim_{k\to\infty} r_k = n$.

Since $S(l,m,n), S(m,l,m)$ and $S(n,m,l)$ are compact so we have $l \in S(l,m,n), m \in S(m,l,m)$ and $n \in S(n,m,l)$.

This shows that (l,m,n) is a tripled fixed point of S.

Using Theorem 2.1, we now prove a tripled coincidence and common fixed point theorems for a hybrid pair of multivalued and single valued mapping.

Theorem 2.2 *Let (X,d) be a complete metric space and let $S: X \times X \times X \to CB(X)$ be a set valued mapping and $f: X \to X$ satisfies*

$H(S(l,m,n),S(a_1,b_1,c_1))$

$$\leq \max\left\{\begin{matrix}\zeta\left(\max(\varsigma(fl,fa_1),\varsigma(fm,fb_1),\varsigma(fn,fc_1))\right), \\ \zeta(\max(\varsigma(\alpha_1,\beta_1),\varsigma(\alpha_2,\beta_2),\varsigma(\alpha_3,\beta_3)))\end{matrix}\right\}$$

$$\begin{bmatrix}\max(\varsigma(fl,fa_1),\varsigma(fm,fb_1),\varsigma(fn,fc_1))) \\ -\max(\varsigma(\alpha_1,\beta_1),\varsigma(\alpha_2,\beta_2),\varsigma(\alpha_3,\beta_3))\end{bmatrix}$$

for some $\alpha_1 \in S(l,m,n), \alpha_2 \in S(m,l,m), \alpha_3 \in S(n,m,l)$ and $\beta_1 \in S(a_1,b_1,c_1), \beta_2 \in S(b_1,a_1,b_1), \beta_3 \in S(c_1,b_1,a_1)$. Where $\varsigma: X \times X \to [0,\infty)$ is lower semi continuous function and $\zeta: [0,\infty) \to [0,\infty)$ be an upper semi continuous function.Further assume that $S(X \times X \times X) \subseteq f(X)$ Then S,f have a tripled coincidence point.

Further, S,f have a tripled common fixed point if one of the following conditions holds.

(a) $\{S,f\}$ is $w-$ compatible, there exists $a_1,b_1,c_1 \in X$ such that $\lim_{k\to\infty} f^k l = a_1, \lim_{k\to\infty} f^k m = b_1$ and $\lim_{k\to\infty} f^k n = c_1$ whenever (l,m,n) is tripled coincidence point of $\{S,f\}$ and f is continuous at a_1,b_1,c_1.

(b) There exists $a_1,b_1,c_1 \in X$ such that $\lim_{k\to\infty} f^k a_1 = l, \lim_{k\to\infty} f^k b_1 = m$ and $\lim_{k\to\infty} f^k c_1 = n$ whenever (l,m,n) is a tripled coincidence point of $\{T,f\}$ and f is continuous at l,m and n.

Proof: By Lemma 1.4, there exists $E \subseteq X$ such that $f: E \to X$ is one to one and $f(E) = f(X)$.

Now define $T: f(E) \times f(E) \times f(E) \to CB(X)$ by $T(fl,fm,fn) = S(l,m,n)$, for all $fl,fm,fn \in f(E)$.

Since f is one-one on E, so T is well defined.

Now

$H(T(fl,fm,fn),T(fa_1,fb_1,fc_1))$

$$= H(S(l,m,n),S(a_1,b_1,c_1))$$

$$\leq \max\left\{\begin{matrix}\zeta\left(\max(\varsigma(fl,fa_1),\varsigma(fm,fb_1),\varsigma(fn,fc_1))\right), \\ \zeta\left(\max(\varsigma(\alpha_1,\beta_1),\varsigma(\alpha_2,\beta_2),\varsigma(\alpha_3,\beta_3))\right)\end{matrix}\right\}$$

$$\begin{bmatrix}\max(\varsigma(fl,fa_1),\varsigma(fm,fb_1),\varsigma(fn,fc_1))) \\ -\max(\varsigma(\alpha_1,\beta_1),\varsigma(\alpha_2,\beta_2),\varsigma(\alpha_3,\beta_3))\end{bmatrix}$$

Hence T satisfies all the conditions and the contraction of Theorem 2.1. So by Theorem 2.1, T has a tripled fixed point say $(u,v,w) \in f(E) \times f(E) \times f(E)$.

Thus,

$$a_1 \in T(a_1,b_1,c_1)$$
$$b_1 \in T(b_1,a_1,b_1)$$
$$c_1 \in T(c_1,b_1,a_1)$$

Since $S(X \times X \times X) \subseteq f(X)$, so there exists $a_2,b_2,c_2 \in X \times X \times X$ such that $fa_2 = a_1, fb_2 = b_1$ and $fc_2 = c_1$.

Now from the above relation we have

$$fa_2 \in T(fa_2,fb_2,fc_2) = S(a_2,b_2,c_2)$$
$$fb_2 \in T(fb_2,fa_2,fb_2) = S(b_2,a_2,b_2)$$
$$fc_2 \in T(fc_2,fb_2,fa_2) = S(c_2,b_2,a_2)$$

This shows that $(a_2,b_2,c_2) \in X \times X \times X$ is a tripled coincidence point of S,f.

Suppose condition (a) holds.

Since (a_2, b_2, c_2) is a tripled coincidence point of T and f, there exists $a_1, b_1, c_1 \in X$ such that $\lim_{k\to\infty} f^k a_2 = a_1, \lim_{k\to\infty} f^k b_2 = b_1$ and $\lim_{k\to\infty} f^k c_2 = c_1$.

Since f is continuous at a_1, b_1 and c_1, we have $fa_1 = a_1, fb_1 = b_1$ and $fc_1 = c_1$.

Since $fa_2 \in S(a_2, b_2, c_2)$, we have $f^2 a_2 \in f(S(a_2, b_2, c_2)) \subseteq S(fa_2, fb_2, fc_2)$.

Since $fb_2 \in S(b_2, a_2, b_2)$, we have $f^2 b_2 \in f(S(b_2, a_2, b_2)) \subseteq S(fb_2, fa_2, fb_2)$.

Since $fc_2 \in S(c_2, b_2, a_2)$, we have $f^2 c_2 \in f(S(c_2, b_2, a_2)) \subseteq S(fc_2, fb_2, fa_1)$.

This shows that (fa_2, fb_2, fc_2) is a tripled coincidence point of T and f.

Similarly, we can prove that $(f^k a_2, f^k b_2, f^k c_2)$ is a tripled coincidence point of T and f.

Therefore we have

$$f^k a_2 \in S(f^{k-1} a_2, f^{k-1} b_2, f^{k-1} c_2)$$
$$f^k b_2 \in S(f^{k-1} b_2, f^{k-1} a_2, f^{k-1} b_2)$$
$$f^k c_2 \in S(f^{k-1} c_2, f^{k-1} b_2, f^{k-1} a_2)$$

Now,

$$d\big(fa_1, S(a_1, b_1, c_1)\big)$$

$$\leq d(fa_1, f^k a_2) + d\big(f^k a_2, S(a_1, b_1, c_1)\big)$$

$$\leq d(fa_1, f^k a_2) + H\big(S(f^{k-1} a_2, f^{k-1} b_2, f^{k-1} c_2), S(a_1, b_1, c_1)\big)$$

$$\leq d(fa_1, f^k a_2)$$

$$+\max\left\{\begin{matrix}\zeta\left(\max\big(\varsigma(f^{k-1}a_2, fa_1), \varsigma(f^{k-1}b_2, fb_1), \varsigma(f^{k-1}c_2, fc_1)\big)\right), \\ \zeta\left(\max\big(\varsigma(\alpha_1, \beta_1), \varsigma(\alpha_2, \beta_2), \varsigma(\alpha_3, \beta_3)\big)\right)\end{matrix}\right\}$$

$$\begin{bmatrix}\max(\varsigma(f^{k-1}a_2, fa_1), \varsigma(f^{k-1}b_2, fb_1), \varsigma(f^{k-1}c_2, fc_1))) \\ -\max(\varsigma(\alpha_1, \beta_1), \varsigma(\alpha_2, \beta_2), \varsigma(\alpha_3, \beta_3))\end{bmatrix}$$

$$\leq d(fa_1, f^k a_2)$$

$$+\max\left\{\begin{matrix}\zeta\left(\max\big(\varsigma(f^{k-1}a_2, fa_1), \varsigma(f^{k-1}b_2, fb_1), \varsigma(f^{k-1}c_2, fc_1)\big)\right), \\ \zeta\left(\max\big(\varsigma(\alpha_1, \beta_1), \varsigma(\alpha_2, \beta_2), \varsigma(\alpha_3, \beta_3)\big)\right)\end{matrix}\right\}$$

$$[\max(\varsigma(f^{k-1}a_2, fa_1), \varsigma(f^{k-1}b_2, fb_1), \varsigma(f^{k-1}c_2, fc_1)))].$$

Letting $k \to \infty$, we obtain

$$d\big(fa_1, S(a_1, b_1, c_1)\big)$$

$$\leq d(fa_1, fa_1)$$

$$+\max\left\{\begin{matrix}\zeta\left(\max\big(\varsigma(fa_1, fa_1), \varsigma(fb_1, fb_1), \varsigma(fc_1, fc_1)\big)\right), \\ \zeta\left(\max\big(\varsigma(\alpha_1, \beta_1), \varsigma(\alpha_2, \beta_2), \varsigma(\alpha_3, \beta_3)\big)\right)\end{matrix}\right\}$$

$$[\max(\varsigma(fa_1, fa_1), \varsigma(fb_1, fb_1), \varsigma(fc_1, fc_1)))]$$

$$\leq 0,$$

which implies that $fa_1 \in S(a_1, b_1, c_1)$.

Thus $a_1 = fa_1 \in S(a_1, b_1, c_1)$. In the same way we can prove that $b_1 = fb_1 \in S(b_1, a_1, b_1), c_1 = fc_1 \in S(c_1, b_1, a_1)$.

This shows that (a_1, b_1, c_1) is a tripled common fixed point of the hybrid pair $\{S, f\}$.

Suppose condition (b) holds.

Since (a_2, b_2, c_2) is a tripled coincidence point of $\{S, f\}$, there exists $a_1, b_1, c_1 \in X$ such that $\lim_{k\to\infty} f^k a_1 = a_2, \lim_{k\to\infty} f^k b_1 = b_2$ and $\lim_{k\to\infty} f^k c_1 = c_2$.

Since f is continuous at a_2, b_2 and c_2,

we have

$fa_2 = a_2, \quad fb_2 = b_2$ and $\quad fc_2 = c_2$.

Thus $a_2 = fa_2 \in S(a_2, b_2, c_2), b_2 = fb_2 = S(b_2, a_2, b_2)$ and $c_2 = fc_2 = S(c_2, b_2, a_2)$.

Hence (a_2, b_2, c_2) is a tripled common fixed point of $\{S, f\}$. Hence the results if proved.

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