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Review Article

OPTIMAL ORDERING POLICY FOR SUBSTITUTABLE PRODUCTS UNDER QUADRATIC DEMAND WITH COST OF SUBSTITUTION

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Abstract
In this paper an inventory system consisting of two products which are mutually substitutable has been considered. When the first product is stocks-out, the demand for it is partially met from the inventory of the second product and vice-versa. During the substitution period, there incur an additional cost of substitution for each unit of the substituted product. The unmet demand is assumed to be lost. The demand of the products is assumed to be quadratic. The two products are ordered together in each ordering cycle. Shortages are allowed. The problem has been formulated to study the impact of cost of substitution on the performance of the inventory system. Solution procedure is discussed to minimize the total cost. Sensitivity analysis has been done to the parameters involved in total cost, to show the substantial improvement in the total cost with substitution compared to the case of without substitution.

Keywords: Product Substitution, Quadratic Demand, Cost of Substitution, Substitution Rate.

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INTRODUCTION

In an inventory system, when a demand of a product is partially or fully met my another product is called product substitution. When substitution occurs there is an additional cost known as substitution cost is involved. There are many reasons for this: the labour charge for reworking on the product to convert it suitable for the other, good will of the customer should not be affected, etc. Note that the product can be fully substituted by another product or partially. There are various reasons for substitution; this event of substitution is widely studied by researches. This article studies an inventory system with two products, which are partially substituted. Also studies the influence of the substitution cost on the functioning of the inventory system.

In deterministic conditions the product substitution happens in a planned manner where as in the case of stochastic the system is uncertain. Drenzer et. Al [2] considered an inventory system with two products under product substitution and studied the difference between the case of no substitution and full substitution. In his paper he found that full substitution is never optimal. Drenzer [4] extended his paper to more than two products. In this case the unmet demand of a product is fully treated as demand for the other product. Salameh et. al [10] extended the paper of Drenzer et. al [4] by considering substitution partially. Krommyda et al. [5] proposed a model where the demand is dependent and stock, which is similar to the work done by Salameh et al [10]. A very interesting aspect about the demand substitution is studied in the case of stochastic demand. In this case substitution becomes alternative under uncertainty. Inventory system with stochastic demand for substitutable product is studied my many researches, major contributions are presented by Xue and Song [11], Ye [12], where they developed a model for multiple products. Whereas, Pasternack and Drenzer [9], and Parlar and Goyal [8] contributed a lot on this model.

The proposed model in this article work in the direction of introduction of substitution cost, the influence of cost of substitution in the functioning of the inventory system.

In this paper, we consider an inventory system with two mutually substitutable products. The demand of the product is assumed to be quadratic. When the first product stocks-out, the demand for it is partially met from the inventory of the second product and the unmet demand of product one is assumed to be lost and vice-versa. There exists an additional cost for substitution, for each unit of the substituted product. The products are ordered together in each ordering cycle. The article is arranged as follows. In Section 2, Assumptions and Notations involved throughout the paper is described. In section 3 the model for the proposed inventory system is presented, the solution procedure is discussed in section 4 showing that the total cost function is pseudo-convex. In section 5 sensitivity analysis has been carried out with some numerical examples. Section 6 submits the conclusions.

Assumptions

a. Let Q_1 and Q_2 be the ordering quantities for product 1 and 2 respectively.
b. Let v_1 and v_2 be the substitution rates for substituting product 1 by product 2 and product 2 by product 1 respectively. The unmet demand of the product is assumed to be lost.
c. Let π_1 and π_2 be the lost sales cost per unit of product 1 and 2 respectively.
d. Let D_1 and D_2 be the demand rates for product 1 and 2, which is assumed to be quadratic.
e. Let k_1 and k_2 be the fixed ordering cost, h_1 and h_2 be the inventory holding cost, c_1 and c_2 be the unit procurement cost for product 1 and 2 respectively.
f. The unit substitution cost for product 1(substituted by product 2) is δ_{12}.

Similarly, δ_{21} be the unit substitution cost for product 2 (substituted by product 1).

g. TCU be the total cost per unit time of the product.

h. Instant replenishment. Lead time is assumed to be zero.

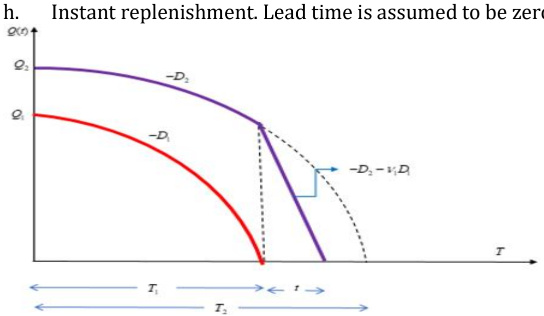

Fig. 1: Inventory system when $T_1 \le T_2$

PROPOSED MODEL

Consider an inventory model consisting of two substitutable products facing quadratic demand. At the beginning there were Q_1 units of product 1 and Q_2 units of product 2. Two cases were studied based on their cycle time.

Case(i): $T_1 \le T_2$

During the interval $[0,t]$ product 1 depletes faster than product 2. When product 1 is out of stock the demand of product 1 is substituted partially from the inventory of product2, with the substitution rate ν_1 and an associated cost of substitution δ_{12} is incurred according as the number of units of product 1 substituted by product 2.

The inventory level of products in $[0,T_1]$ is given by

$$Q = Q_2 - \frac{D_1^2}{Q_1}T^2, 0 < T < T_1 \qquad (1)$$

$$Q = Q_2 - \frac{D_2^2}{Q_2}T^2, \ 0 < T < T_1 \qquad (2)$$

During the interval $[T_1,t]$ substitution takes place for product 1 by product 2. The inventory level in this period is given by

$$Q = (Q_1 + Q_2) - (D_2 + \nu_1 D_1)T \text{ , } T_1 < T < t \quad (3)$$

The total cost for case (i) consists of the following components:

a) Ordering cost

For product 1 = k_1

For product 2= k_2

b) Purchase cost

For product 1= $c_1 Q_1$

For product 2= $c_2 Q_2$

c) Inventory holding cost during the interval $[0,T_1]$

For product 1= $\dfrac{2}{3}\dfrac{h_1 c_1 Q_1^2}{D_1}$

For product 2= $\dfrac{2}{3}\dfrac{h_2 c_2 Q_2^2}{D_2}$

d) Inventory holding cost during the interval $[T_1,t]$

IH= $\dfrac{D_1^2 c_2 h_2 (Q_1+Q_2)^2 (1-\nu_1)^2}{2(D_2+\nu_1 D_1)(D_1+D_2)^2}$

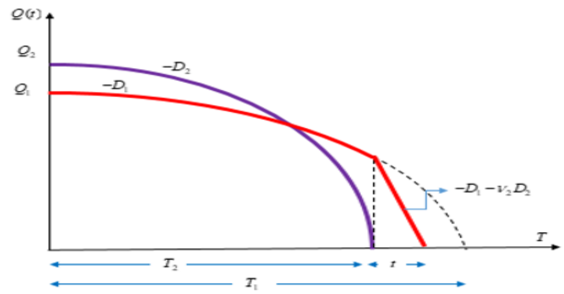

Fig. 2: Inventory system when $T_1 \ge T_2$

e) Lost sales cost = $\dfrac{D_1^2 \pi_1 (Q_1+Q_2)(1-\nu_1)^2}{(D_2+\nu_1 D_1)(D_1+D_2)}$

f) Substitution cost= $\delta_{12} D_1 \nu_1 \left(\dfrac{D_1 (Q_1+Q_2)(1-\nu_1)}{(D_2+\nu_1 D_1)(D_1+D_2)} \right)$

The total cost $TC_1(Q_1,Q_2)$ for case (i) is obtained by adding up all the above components

$$TC_1 = k_1 + k_2 + c_1 Q_1 + c_2 Q_2 + \frac{2}{3}\frac{h_1 c_1 Q_1^2}{D_1} + \frac{2}{3}\frac{h_2 c_1 Q_2^2}{D_2} + \frac{D_1^2 c_2 h_2 (Q_1+Q_2)^2 (1-\nu_1)^2}{2(D_2+\nu_1 D_1)(D_1+D_2)^2} + \frac{D_1^2 \pi_1 (Q_1+Q_2)(1-\nu_1)^2}{(D_2+\nu_1 D_1)(D_1+D_2)} + \delta_{12} D_1 \nu_1 \left(\frac{D_1 (Q_1+Q_2)(1-\nu_1)}{(D_2+\nu_1 D_1)(D_1+D_2)} \right) \qquad (4)$$

The total cost per unit time $TCU_1(Q_1, Q_2)$ is obtained by dividing the above total cost equation by the cycle time $t = \left(\frac{Q_1+Q_2}{D_2+\nu_1 D_1}\right)$ which is given as

$$TCU_1 = \left(\frac{D_2+\nu_1 D_1}{Q_1+Q_2}\right)\left[k_1+k_2+c_1Q_1+c_2Q_2+\frac{2}{3}\frac{h_1c_1Q_1^2}{D_1}+\frac{2}{3}\frac{h_2c_1Q_2^2}{D_2}+\frac{D_1^2c_2h_2\left(Q_1+Q_2\right)^2\left(1-\nu_1\right)^2}{2\left(D_2+\nu_1D_1\right)\left(D_1+D_2\right)^2}+\frac{D_1^2\pi_1\left(Q_1+Q_2\right)\left(1-\nu_1\right)^2}{\left(D_2+\nu_1D_1\right)\left(D_1+D_2\right)}+\delta_{12}D_1\nu_1\left(\frac{D_1\left(Q_1+Q_2\right)\left(1-\nu_1\right)}{\left(D_2+\nu_1D_1\right)\left(D_1+D_2\right)}\right)\right] \quad (5)$$

Case (ii): $T_1 \geq T_2$

In this case product 2 stocks out first, then a partial demand of product 2 is substituted by product 1 with the substitution rate ν_2 and cost of substitution δ_{21} per unit of product 2 substituted by product 1. Substitution cost is considered according to the number of units of product 2 substituted by product 1.

The total cost $TC_2(Q_1, Q_2)$ equation is derived similarly as case(i) is given by

$$TC_2 = k_1+k_2+c_1Q_1+c_2Q_2+\frac{2}{3}\frac{h_1c_1Q_1^2}{D_1}+\frac{2}{3}\frac{h_2c_1Q_2^2}{D_2}+\frac{D_1^2c_1h_1\left(Q_1+Q_2\right)^2\left(1-\nu_2\right)^2}{2\left(D_1+\nu_2D_2\right)\left(D_1+D_2\right)^2}+\frac{D_1^2\pi_1\left(Q_1+Q_2\right)\left(1-\nu_2\right)^2}{\left(D_1+\nu_2D_2\right)\left(D_1+D_2\right)}+\delta_{21}D_1\nu_2\left(\frac{D_1\left(Q_1+Q_2\right)\left(1-\nu_2\right)}{\left(D_1+\nu_2D_2\right)\left(D_1+D_2\right)}\right) \quad (6)$$

Similarly, the total cost per unit time $TCU_2(Q_1, Q_2)$ is obtained as

$$TCU_2 = \left(\frac{D_1+\nu_2 D_2}{Q_1+Q_2}\right)\left[k_1+k_2+c_1Q_1+c_2Q_2+\frac{2}{3}\frac{h_1c_1Q_1^2}{D_1}+\frac{2}{3}\frac{h_2c_1Q_2^2}{D_2}+\frac{D_1^2c_1h_1\left(Q_1+Q_2\right)^2\left(1-\nu_2\right)^2}{2\left(D_1+\nu_2D_2\right)\left(D_1+D_2\right)^2}+\frac{D_1^2\pi_1\left(Q_1+Q_2\right)\left(1-\nu_2\right)^2}{\left(D_1+\nu_2D_2\right)\left(D_1+D_2\right)}+\delta_{21}D_1\nu_2\left(\frac{D_1\left(Q_1+Q_2\right)\left(1-\nu_2\right)}{\left(D_1+\nu_2D_2\right)\left(D_1+D_2\right)}\right)\right] \quad (7)$$

Solution Procedure*:*

To minimize the total cost in case (i) and (ii), We state and prove the following theorem.

Theorem:1.1 The total cost $TCU_1(Q_1, Q_2)$ is pseudo convex function.

Proof: To prove this theorem, we first claim that $TC_1(Q_1, Q_2)$ is convex.

we have

$$\frac{\partial^2 TC_1}{\partial Q_1^2} = \frac{4c_1h_1}{3D_1}+\frac{2D_1^2c_2h_2\left(1-\nu_1\right)^2}{2\left(D_2+\nu_1D_1\right)\left(D_1+D_2\right)^2} \geq 0$$

and

$$\frac{\partial^2 TC_1}{\partial Q_2^2} = \frac{4c_2h_2}{3D_2}+\frac{2D_1^2c_2h_2\left(1-\nu_1\right)^2}{2\left(D_2+\nu_1D_1\right)\left(D_1+D_2\right)^2} \geq 0$$

Also $\frac{\partial^2 TC_1}{\partial Q_1 \partial Q_2} = \frac{2D_1^2c_2h_2\left(1-\nu_1\right)^2}{2\left(D_2+\nu_1D_1\right)\left(D_1+D_2\right)^2} \geq 0$

To complete the proof of the claim, we note that the determinant of the Hessian matrix

$H = \left(\frac{\partial^2 TC_1}{\partial Q_1^2} \quad \frac{\partial^2 TC_1}{\partial Q_2^2}\right) - \left(\frac{\partial^2 TC_1}{\partial Q_1 \partial Q_2}\right)^2 > 0$. I.e. It is positive semi-definite.

Hence our claim $TC_1(Q_1, Q_2)$ is convex is true.

Now to prove the statement of the theorem, we make use of the fact that ratio of positive convex function over a linear function is pseudo convex. Therefore, the total cost per unit time is pseudo convex.

Theorem:1.2 The total cost $TCU_2(Q_1, Q_2)$ is pseudo convex function

Proof: Similar to the proof of Theorem 1.1

We now state an algorithm to obtain the optimal ordering quantity.

Step 1: Initialize the values of the parameters in the proposed model.

Step 2: Solve the optimization problem $\min_{Q_1, Q_2} TCU$ subject to $\frac{Q_1}{D_1} \le \frac{Q_2}{D_2}$ and the obtain the optimal solution of ordering quantity in case (i) be $\left(Q_{11}^*, Q_{21}^*\right)$ and the optimal total cost is TCU_1^*.

Step 3: Solve the optimization problem $\min_{Q_1, Q_2} TCU$ subject to $\frac{Q_1}{D_1} \ge \frac{Q_2}{D_2}$ and the obtain the optimal solution of ordering quantity in case (ii) be $\left(Q_{12}^*, Q_{22}^*\right)$ and the optimal total cost is TCU_2^*.

Step 4: Compare TCU_1^* and TCU_2^*. Obtain the optimal ordering quantity as $\left(Q_1^*, Q_2^*\right)$.

Sensitivity Analysis

In this section, we take a numerical example to illustrate the proposed model. For initialization, we choose the value of the parameter as follows:

$$D_1=200; D_2=100; h_1=10; h_2=20; k_1=150; k_2=200; \pi_1=1; c_1=3; c_2=5; \delta_{12}=2; \nu_1=0.2;$$

The algorithm has been used for the given initial parameter values by using MATLAB, on solving Step 1 we obtain optimal ordering quantity for case (i) is $\left(Q_{11}^*, Q_{21}^*\right) = \left(26.58, 13.29\right)$ and the optimal total cost is $TCU_1^* = 3098.91$. Now solving Step 2, we obtain the optimal ordering quantity for case (ii) is $\left(Q_{12}^*, Q_{22}^*\right) = \left(34.79, 17.39\right)$ and the optimal total cost is $TCU_2^* = 4211.28$. On comparing, we see that $TCU_1^* < TCU_2^*$. Therefore the optimal ordering policy is $\left(Q_{11}^*, Q_{21}^*\right) = \left(26.58, 13.29\right)$. Now we consider the case of without substitution. The total cost per unit time is given as

$$TCU_{WOS} = \left(\frac{D_1 + D_2}{Q_1 + Q_2}\right)\left[k_1 + k_2 + c_1 Q_1 + c_2 Q_2 + \frac{2}{3}\frac{h_1 c_1 Q_1^2}{D_1} + \frac{2}{3}\frac{h_2 c_2 Q_2^2}{D_2}\right]$$

The optimal value is obtained as $\left(Q_1^*, Q_2^*\right) = \left(36.23, 18.11\right)$ and the optimal total cost is $TCU_2^* = 4964.37$ using MATLAB.

We could see clearly on comparison between the case of substitution over without substitution, there is a cost difference of Rs.1865.45 and 37.58% of improvement in the total cost. Next, sensitivity analysis has been done for various parameters in the proposed model. The improvement in percentage of total cost is presented in Table 1. The results are presented by following figures. The substantial improvement of substitution over without substitution is very reasonable.

Table 1: Sensitivity analysis of parameters

Parameter	Values	with substitution			without substitution			
		Q_1	Q_2	TCU	Q_1	Q_2	TCU	%Improvement
ν_1	0.15	25.35	12.68	2999.96	36.23	18.11	4964.37	39.57
	0.25	27.76	13.88	3196.57	36.23	18.11	4964.37	35.61
	0.35	29.92	14.96	3391.71	36.23	18.11	4964.37	31.68
	0.5	32.61	16.30	3695.60	36.23	18.11	4964.37	25.56
h_1	2	29.03	14.52	2891.68	43.30	21.65	4333.16	33.27
	4	28.36	14.18	2945.24	41.15	20.58	4501.96	34.58
	6	27.73	13.86	2997.58	39.30	19.65	4662.77	35.71
	8	27.14	13.57	3048.78	37.67	18.83	4816.63	36.70
π_1	1	26.58	13.29	3098.91	36.23	18.11	4964.37	37.58
	5	26.58	13.29	3440.25	36.23	18.11	4964.37	30.70
	8	26.58	13.29	3696.25	36.23	18.11	4964.37	25.54
	10	26.58	13.29	3866.91	36.23	18.11	4964.37	22.11
k_1	125	25.62	12.81	3009.52	34.91	17.46	4823.80	37.61
	155	26.77	13.39	3116.41	36.49	18.24	4991.87	37.57
	185	27.88	13.94	3218.87	38.00	19.00	5152.98	37.53
	215	28.95	14.47	3317.41	39.45	19.72	5307.93	37.50

δ_{12}	2	26.58	13.29	3098.91	36.23	18.11	4964.37	37.58
	4	26.58	13.29	3141.58	36.23	18.11	4964.37	36.72
	6	26.58	13.29	3184.25	36.23	18.11	4964.37	35.86
	8	26.58	13.29	3226.91	36.23	18.11	4964.37	35.00
c_1/c_2	0.20	63.90	31.95	1215.75	93.54	46.77	1636.66	25.72
	0.40	61.55	30.78	1273.43	86.60	43.30	1796.58	29.12
	0.60	59.44	29.72	1329.73	81.01	40.50	1948.20	31.75
	0.80	57.54	28.77	1384.78	76.38	38.19	2093.03	33.84

In Table 2, all the findings have been listed. The objective of minimizing the total cost when there is a cost of substitution is involved is in detail analysed. Many organizations thereby increase the performance of their inventory substantially through product substitution.

Table 2: Improvement in optimal total cost

Parameters	Variation	TCU(With substitution)	TCU(without substitution)	% Improvement
v_1	Increases	Increases	Constant	Decrease
h_1	Increases	Increases	Increases	Increases
π_1	Increases	Increases	Constant	Decrease
k_1	Increases	Increases	Increases	Increases
δ_{12}	Increases	Increases	Constant	Decrease
c_1/c_2	Increases	Increases	Increases	Increases

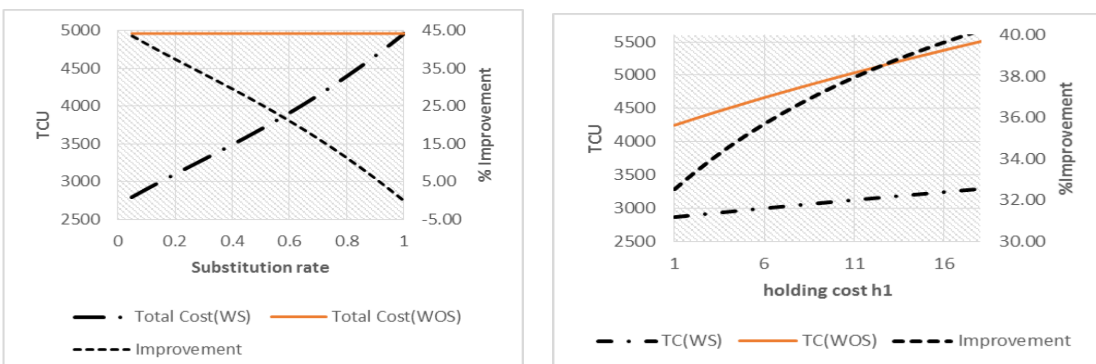

Fig. 3: % Improvement of TCU with substitution over without substitution as v_1 and h_1 increases

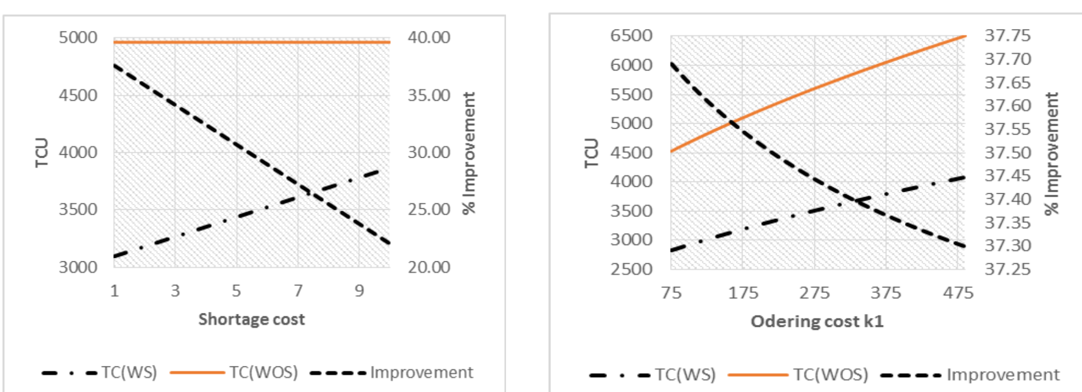

Fig. 4: % Improvement of TCU with substitution over without substitution as π_1 and k_1 increases

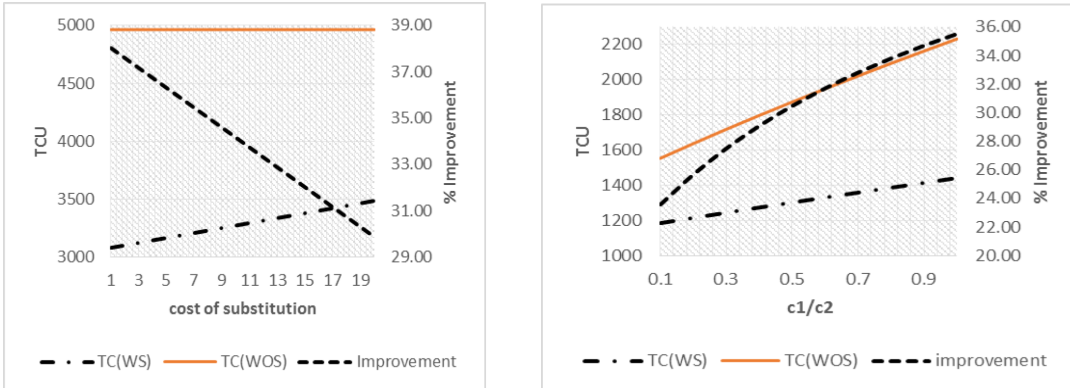

Fig. 5: % Improvement of TCU with substitution over without substitution as δ_{12} and c_1/c_2 increases

CONCLUSION

In this article an inventory system with two mutually substitutable products was considered. Each product faces their own demand, which is assumed to be quadratic. When one product stocks out, the demand of it is partially met from the inventory of the second product. When doing so, there is an additional cost known as substitution cost is included in the total cost for each unit that is substituted. The model has been formulated; a solution procedure is discussed through calculus for obtaining the optimal total cost. Sensitivity analysis has been carried out for different parameter in the model. Numerical results show that there is substantial improvement in the total cost with product substitution over without substitution. When the substitution cost increases, percentage improvement decreases, therefore it is no longer beneficial to do substitution when the cost of substitution is way beyond. Further extension of this model can be studied for more than two products.

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