Perfect Dominating sets and Perfect Domination Polynomial of a Pan Graph

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ABSTRACT: Let $G = (V, E)$ be a simple graph. A set $S \subseteq V$ is a perfect dominating set of G, if every vertex u in $V - S$ is adjacent to exactly one vertex in S. Let $\wp_{pf}(n,i)$ be the family of perfect dominating sets of a Pan Graph $\mathcal{P}_n$ with cardinality i, let $d_{pf}(\mathcal{P}_n, i) = |\wp_{pf}(n,i)|$. In this paper, we construct $\wp_{pf}(n,i)$ for a Pan Graph $\mathcal{P}_n$ with $n+1$ vertices and obtain a recursive formula for $d_{pf}(\mathcal{P}_n, i)$. The polynomial $D_{pf}(\mathcal{P}_n, x) = \sum_{i=\gamma_{pf}(\mathcal{P}_n)}^{n} d_{pf}(\mathcal{P}_n, i)\, x^i$ is also considered using this recursive formula, which we call the perfect domination polynomial of a Pan Graph, which further aids in the study of some properties of this polynomial.

KEYWORDS: Perfect Dominating set, Pan Graph.

I. INTRODUCTION

Let $G = (V, E)$ be a simple graph of order $|V| = n$. For any vertex $u \in V$, the open neighborhood of u is the set $N(u) = \{v \in V | uv \in E\}$. A set $S \subseteq V$ is a dominating set of G, if every vertex $u \in V$ is a element of S or is adjacent to an element of S [7]. The dominating set S is a perfect dominating set if $|N(u) \cap S| = 1$ for each $u \in V - S$, or equivalently, if every vertex u in $V - S$ is adjacent to exactly one vertex in S [8]. The perfect domination number γ_{pf} is the minimum cardinality of a perfect dominating set in G. The Pan Graph is the graph obtained by joining a cycle graph C_n to a singleton graph K_1 with a bridge[5]. The Pan Graph is denoted by $\mathcal{P}_n$. Let $\mathcal{P}_n$ be a Pan Graph with $n+1$ vertices. Let $\wp_{pf}(n,i)$ be the family of perfect dominating sets of a Pan Graph $\mathcal{P}_n$ with cardinality i, let $d_{pf}(\mathcal{P}_n, i) = |\wp_{pf}(n,i)|$. We call the polynomial $D_{pf}(\mathcal{P}_n, x) = \sum_{i=\gamma_{pf}(\mathcal{P}_n)}^{n} d_{pf}(\mathcal{P}_n, i)\, x^i$, as a perfect domination polynomial of a Pan Graph $\mathcal{P}_n$. We use $\lceil n \rceil$, for the smallest integer greater than or equal to n. In this paper $[n]$ denotes the set $\{1,2,\dots,n\}$.

II. Perfect Dominating sets of a Pan Graph

Let $\wp_{pf}(n,i)$ be the family of perfect dominating sets of the Pan Graph $\mathcal{P}_n$ with cardinality i. The perfect dominating sets of the Pan Graph $\mathcal{P}_n$ is observed here.

The following lemmas are essential to prove the main result in this paper.

Lemma 2.1

$i)\ \gamma_{pf}(\mathcal{P}_{3n}) = n$ for $n \in N$

$ii)\ \gamma_{pf}(\mathcal{P}_{3n+1}) = \left\lceil \frac{3n+1}{3} \right\rceil$ for $n \in N$

$iii)\ \gamma_{pf}(\mathcal{P}_{3n+2}) = n + 2$ for $n \in N$

By Lemma 2.1 and the definition of perfect domination number, we have the following lemma.

Lemma 2.2

For $n \geq 3$, $\wp_{pf}(n,i) = \emptyset$, if and only if $i > n+1$

Proof

Let $\mathscr{p}_n$ be a Pan Graph with $n+1$ vertices and any member of $\wp_{pf}(n,i)$ contains atmost $n+1$ vertices.

Therefore, $\wp_{pf}(n,i) = \emptyset$ for $i > n+1$.

Conversely, if $i > n+1$ then by definition of perfect dominating set of a Pan Graph $\mathscr{p}_n$ we have $\wp_{pf}(n,i) = \emptyset$.

Lemma 2.3

For $n \geq 3$,

$i)$ $\wp_{pf}(n,n) = \{1,2,3,\dots,n\}$

$ii)$ $\wp_{pf}(n,n+1) = \{[n+1]\}$

$iii)$ $\wp_{pf}(n,n-1) = \{[n] - \{i,i+1\} | i = 1,2,3,\dots n-2\}$

Proof

$i)$ By the Definition of Perfect dominating Set if we choose n vertices from $n+1$ vertices the only Possible set is $\{1,2,3,\dots,n\}$

$ii)$ We know that for any $G = (V,E)$, $V(G)$ is always a perfect dominating set of G. Hence, $\wp_{pf}(n,n+1) = \{[n+1]\}$

$iii)$ Here, $\wp_{pf}(n,n-1)$ is the family of perfect dominating sets with cardinality $n-1$ in a Pan Graph of $n+1$ vertices. Clearly, $\{[n+1] - \{i,i+1\} | i = 1,2,3,\dots n-2\}$ are the possible perfect dominating sets with cardinality $n-1$.

Lemma 2.4

For $n \in N$

$i)$ $\wp_{pf}(3n,n) = \{3,6,\dots,3n\}$

$ii)$ $\wp_{pf}(3n,n+1) = \{3,6,\dots,3n,3n+1\}$

$iii)$ $\wp_{pf}(3n+2,n+1) = \emptyset$

Lemma 2.5

If $\wp_{pf}(n,i)$ be a family of perfect dominating sets with cardinality i then, $|\wp_{pf}(n,i)| = |\wp_{pf}(n-1,i-1)| + |\wp_{pf}(n-3,i-1)|$

III. Perfect Domination Polynomial of a Pan Graph

Definition 3.1

Let $\mathscr{p}_n$ be a Pan Graph with $n+1$ vertices. Let $\wp_{pf}(n,i)$ be the family of perfect dominating sets of the Pan Graph with cardinality i and $d_{pf}(\mathscr{p}_n,i) = |\wp_{pf}(n,i)|$. Then the Perfect dominating polynomial of the Pan Graph $\mathscr{p}_n$ is given by $D_{pf}(\mathscr{p}_n,x) = \sum_{i=\gamma_{pf}(\mathscr{p}_n)}^{n} d_{pf}(\mathscr{p}_n,i)\, x^i$.

Theorem 3.2

If $\wp_{pf}(n,i)$ be a family of perfect dominating sets with cardinality i then,

For every $n \geq 6$, $D_{pf}(\mathscr{p}_n,x) = x[D_{pf}(\mathscr{p}_{n-1},x) + D_{pf}(\mathscr{p}_{n-3},x)]$ with initial values $D_{pf}(\mathscr{p}_3,x) = x + x^2 + x^3 + x^4$, $D_{pf}(\mathscr{p}_4,x) = 3x^2 + 2x^3 + x^4 + x^5$, $D_{pf}(\mathscr{p}_5,x) = 4x^3 + 3x^4 + x^5 + x^6$.

Proof

It follows from the definition of perfect domination polynomial and Lemma 2.6

Using the above theorem we obtain $d_{pf}(\wp_n, i)$ for $3 \le n \le 15$ as shown in the following **Table 1**

i \ n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
3	1	1	1	1												
4	0	3	2	1	1											
5	0	0	4	3	1	1										
6	0	1	1	5	4	1	1									
7	0	0	4	3	6	5	1	1								
8	0	0	0	8	6	7	6	1	1							
9	0	0	1	1	13	10	8	7	1	1						
10	0	0	0	5	4	19	15	9	8	1	1					
11	0	0	0	0	13	10	26	21	10	9	1	1				
12	0	0	0	1	1	26	20	34	28	11	10	1	1			
13	0	0	0	0	6	5	45	35	43	36	12	11	1	1		
14	0	0	0	0	0	19	15	71	56	53	45	13	12	1	1	
15	0	0	0	0	1	1	45	35	105	84	64	55	14	13	1	1

Table 1

Example 3.3

Consider the Pan Graph $\wp_6$ with 7 vertices. We construct a perfect dominating polynomial $D_{pf}(\wp_6, x)$ using **Theorem 3.2** & **Table 1**

Then the Perfect dominating polynomial of a Pan Graph $\wp_6$ is given by $D_{pf}(\wp_6, x) = \sum_{i=\gamma_{pf}(\wp_6)}^{6} d_{pf}(\wp_6, i)\, x^i$ From the **Table 1** we have $d_{pf}(\wp_6, 2) = 1$, $d_{pf}(\wp_6, 3) = 1$, $d_{pf}(\wp_6, 4) = 5$, $d_{pf}(\wp_6, 5) = 4$, $d_{pf}(\wp_6, 6) = 1$, $d_{pf}(\wp_6, 7) = 1$ Hence, $D_{pf}(\wp_6, x) = x^2 + x^3 + 5x^4 + 4x^5 + x^6 + x^7$.

Theorem 3.4

The coefficients of $D_{pf}(\wp_n, x)$ have the following properties

$i)$ $d_{pf}(\wp_n, n) = 1$ for every $n \ge 3$

$ii)$ $d_{pf}(\wp_n, n+1) = 1$ for every $n \ge 3$

$iii)$ $d_{pf}(\wp_{3n}, n) = 1$ for every $n \in N$

$iv)$ $d_{pf}(\wp_{3n}, n+1) = 1$ for every $n \in N$

$v)$ $d_{pf}(\wp_n, n-1) = n-2$ for every $n \ge 3$

$vi)$ $d_{pf}(\wp_n, n-3) = \frac{n^2-9n+20}{2}$ for every $n \ge 6$

$vii)$ $d_{pf}(\wp_n, n-2) = n-1$ for every $n \ge 4$

$viii)$ $d_{pf}(\wp_n, n-4) = \frac{n^2-7n+8}{2}$ for every $n \ge 6$

$ix)$ $d_{pf}(\wp_{3n+2}, n+1) = 0$ for every $n \ge 1$

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$x)\ d_{pf}(\wp_n, n-6) = \frac{n^3-18n^2+95n-120}{6}$ for every $n \geq 9$

Proof

(i),(ii), *(iii)* and (iv) follows from lemma 2.3:*(i),(ii)* and lemma 2.4:*(i),(ii)* respectively

$v)$ We prove this result by induction on n. The result is true for $n = 3$. Since, $\wp_{pf}(3,2) = \{3,4\}$. So, $d_{pf}(\wp_3, 2) = |\wp_{pf}(3,2)| = 1$.

Now, assume that the result is true for all natural numbers less than n. For n, we have by lemma 2.5 $d_{pf}(\wp_n, n-1) = d_{pf}(\wp_{n-1}, n-2) + d_{pf}(\wp_{n-3}, n-2)$.

By induction hypothesis we have, $d_{pf}(\wp_{n-1}, n-2) = n-3$.

Also by part (ii) we have $d_{pf}(\wp_{n-3}, n-2) = 1$.

Therefore, $d_{pf}(\wp_n, n-1) = n-3+1 = n-2$

$vi)$ We prove this result by induction on n. The result is true for $n = 6$. Since, $\wp_{pf}(6,3) = \{3,6,7\}$. So, $d_{pf}(\wp_6, 3) = |\wp_{pf}(6,3)| = 1$.

Now, assume that the result is true for all natural numbers less than n. For n, we have by lemma 2.5 $d_{pf}(\wp_n, n-3) = d_{pf}(\wp_{n-1}, n-4) + d_{pf}(\wp_{n-3}, n-4)$.

By induction hypothesis we have, $d_{pf}(\wp_{n-1}, n-4) = \frac{n^2-11n+30}{2}$.

Also by part (v) we have $d_{pf}(\wp_{n-3}, n-4) = n-5$.

Therefore, $d_{pf}(\wp_n, n-3) = \frac{n^2-9n+20}{2}$ for every $n \geq 6$

$vii)$ We prove this result by induction on n. The result is true for $n = 4$. Since, $\wp_{pf}(4,2) = \{\{1,5\},\{2,4\},\{3,4\}\}$. So, $d_{pf}(\wp_4, 2) = |\wp_{pf}(4,2)| = 3$.

Now, assume that the result is true for all natural numbers less than n. For n, we have by lemma 2.5 $d_{pf}(\wp_n, n-2) = d_{pf}(\wp_{n-1}, n-3) + d_{pf}(\wp_{n-3}, n-3)$.

By induction hypothesis we have, $d_{pf}(\wp_{n-1}, n-3) = n-2$.

Also by part (i) we have $d_{pf}(\wp_{n-3}, n-3) = 1$.

Therefore, $d_{pf}(\wp_n, n-2) = n-2+1 = n-1$ for every $n \geq 4$

$viii)$ We prove this result by induction on n. The result is true for $n = 6$. Since, $\wp_{pf}(6,3) = \{3,6\}$. So, $d_{pf}(\wp_6, 2) = |\wp_{pf}(6,2)| = 1$.

Now, assume that the result is true for all natural numbers less than n. For n, we have by lemma 2.5 $d_{pf}(\wp_n, n-3) = d_{pf}(\wp_{n-1}, n-5) + d_{pf}(\wp_{n-3}, n-5)$.

By induction hypothesis we have, $d_{pf}(\wp_{n-1}, n-5) = \frac{n^2-9n+16}{2}$.

Also by part (vii) we have $d_{pf}(\wp_{n-3}, n-5) = n-4$.

Therefore, $d_{pf}(\wp_n, n-3) = \frac{n^2-7n+8}{2}$ for every $n \geq 6$

$ix)$It follows from lemma 2.4*(iii)*

$x)$ We prove this result by induction on n. The result is true for $n = 9$. Since, $\wp_{pf}(9,3) = \{3,6,9\}$. So, $d_{pf}(\wp_9, 3) = |\wp_{pf}(9,3)| = 1$.

Now, assume that the result is true for all natural numbers less than n. For n, we have by lemma 2.5 $d_{pf}(\wp_n, n-6) = d_{pf}(\wp_{n-1}, n-7) + d_{pf}(\wp_{n-3}, n-7)$.

By induction hypothesis we have, $d_{pf}(\wp_{n-1}, n-7) = \frac{n^3-21n^2+134n-234}{6}$.

Also by part $(viii)$ we have $d_{pf}(\wp_{n-3}, n-7) = \frac{n^2-13n+38}{2}$.

Therefore, $d_{pf}(\wp_n, n-6) = \frac{n^3-18n^2+95n-120}{6}$ for every $n \geq 9$

Theorem 3.5

For every $n \in N$ and $\gamma_{pf}(\wp_n) \leq i \leq n+1$. $|\wp_{pf}(n,i)|$ is the coefficient of $u^n v^i$ in the expansion of the function $f(u,v) = \frac{u^3v(1+uv+uv^2+u^3v+uv^3+uv^4+u^3v^2+u^3v^3+u^3v^4)}{(1-uv-u^3v)}$

Proof

First we set $f(u,v) = \sum_{n=3}^{\infty}\sum_{i=1}^{\infty}|\wp_{pf}(n,i)|u^n v^i$. By using the recursive formula for $|\wp_{pf}(n,i)|$ we can write $f(u,v)$ as follows:

$$f(u,v) = \sum_{n=3}^{\infty}\sum_{i=1}^{\infty}\left(|\wp_{pf}(n-1,i-1)| + |\wp_{pf}(n-3,i-1)|\right)u^n v^i$$

$$= uv\sum_{n=3}^{\infty}\sum_{i=1}^{\infty}|\wp_{pf}(n-1,i-1)|u^{n-1}v^{i-1} + u^3v\sum_{n=3}^{\infty}\sum_{i=1}^{\infty}|\wp_{pf}(n-3,i-1)|u^{n-3}v^{i-1}$$

$$= uv\big(|\wp_{pf}(2,0)|u^2 + |\wp_{pf}(2,1)|u^2v + |\wp_{pf}(2,2)|u^2v^2 + |\wp_{pf}(2,3)|u^2v^3 + |\wp_{pf}(3,0)|u^3 + |\wp_{pf}(3,1)|u^3v + |\wp_{pf}(3,2)|u^3v^2 + |\wp_{pf}(3,3)|u^3v^3 + |\wp_{pf}(3,4)|u^3v^4\big) + uvf(u,v) + u^3v\big(|\wp_{pf}(0,0)| + |\wp_{pf}(1,0)|u + |\wp_{pf}(1,1)|uv + |\wp_{pf}(1,2)|uv^2 + |\wp_{pf}(2,0)|u^2 + |\wp_{pf}(2,1)|u^2v + |\wp_{pf}(2,2)|u^2v^2 + |\wp_{pf}(2,3)|u^2v^3 + |\wp_{pf}(3,0)|u^3 + |\wp_{pf}(3,1)|u^3v + |\wp_{pf}(3,2)|u^3v^2 + |\wp_{pf}(3,3)|u^3v^3 + |\wp_{pf}(3,4)|u^3v^4\big) + u^3vf(u,v).$$

Substituting the values from **Table 1** also for $|\wp_{pf}(n,0)| = 0$ for all $n \in N$ and $|\wp_{pf}(0,0)| = 1$ we have,

$$f(u,v) = uv(u^3v + u^3v^2 + u^3v^3 + u^3v^4) + uvf(u,v) + u^3v(1 + u^3v + u^3v^2 + u^3v^3 + u^3v^4) + u^3vf(u,v)$$

$$f(u,v) = u^3v(1 + u^3v + u^3v^2 + u^3v^3 + u^3v^4 + uv + uv^2 + uv^3 + uv^4) + uvf(u,v) + u^3vf(u,v)$$

$$f(u,v)(1 - uv - u^3v) = u^3v(1 + u^3v + u^3v^2 + u^3v^3 + u^3v^4 + uv + uv^2 + uv^3 + uv^4)$$

Hence, $f(u,v) = \frac{u^3v(1+uv+uv^2+u^3v+uv^3+uv^4+u^3v^2+u^3v^3+u^3v^4)}{(1-uv-u^3v)}$

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