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On the Oscillation of Impulsive Vector Partial Conformable Fractional Differential Equations

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Abstract

In this paper, we consider a new class of oscillation criteria for impulsive vector partial conformable fractional differential equations with continuous distributed deviating arguments. We derive sufficient conditions for the H-oscillation of the solutions, using impulsive differential inequalities and the averaging technique with the Dirichlet boundary condition. We go through various examples to illustrate the improvement achieved by the results which complement and extend those established for problems without impulses.

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1. Introduction

Originally, the theory of fractional derivatives was developed as a field of mathematics, primarily, meant for theoretical study. However, in the last few decades many authors have pointed out that fractional derivatives and fractional integrals are well suited to describe features of real world problems. The mathematical modeling and simulation of systems and processes, based on the description of their properties by fractional derivatives, naturally, leads to differential equations of fractional order and the need to solve them. Fractional differential equations are generalizations of the classical differential equations of integer order and have gained popularity and importance over the last three decades, mainly, due to their documented applications in numerous seemingly diverse and widespread fields of science and engineering. Nowadays the number of scientific and engineering problems involving fractional calculus is already very large and still growing. We find out that several interdisciplinary applications can elegantly be solved with the help of the fractional derivatives. Fractional differentials and integrals enable to build more accurate models of the systems under consideration. Current areas of application of the fractional calculus include fluid flow, rheology, dynamical processes in self-similar and porous structures, diffusive transport akin to diffusion, electrical networks, probability and statistics, control theory of dynamical systems, viscoelasticity, electro-chemistry of corrosion, chemical physics, optics, and signal processing, economics, and so on. Recently, there have been several books on the subject of fractional derivatives and fractional integrals, such as [12, 22]. The obtained fractional derivatives in this calculus have seemed complicated and lost some of the basic properties of the usual derivatives, such as the product rule and the chain rule. But in 2014 Khalil et. al [11] introduced a new fractional derivative called the conformable derivative which closer resembles the classical derivative. In recent years, many researchers have found that fractional differential equations more accurately describe real world systems and phenomena.

The oscillation theory of impulsive differential equations, which provides a natural description of observed evolution processes, is regarded as an important mathematical tool for better understanding of several real world problems in the applied sciences. For more detailed discussions on the applications of impulsive differential equations, we refer the reader to the monographs [14, 15, 25, 27] and reference cited therein.

In 1970, Dom$\check{s}lak$ introduced the concept of H-oscillation to study the oscillation of the solutions of vector differential equations, where H is a unit vector in $\mathbb{R}^M$. We refer the reader to [7, 8, 13] for vector ordinary differential equations, [17-21] for vector partial differential equations and [3, 6, 16, 23] for impulsive vector partial differential equations. In the last decades, the oscillation of fractional differential equations as a new research filed has received significant attention and some interesting results have already been obtained. Some notable results in this

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field can be found in [2, 4, 5, 10, 24] and the references cited therein.

To the best of the authors' knowledge, there has been no work dealing with impulsive vector partial conformable fractional differential equations.

Motivated by this scarcity, we proceed to study the following boundary value problem

$$\begin{cases} \frac{\partial^\alpha}{\partial t^\alpha}\left[r(t)\frac{\partial^\alpha}{\partial t^\alpha}(U(x,t))\right] + \int_c^d q(x,t,\xi)f(U(x,\theta(t,\xi)))d\eta(\xi) \\ = a(t)\Delta U(x,t) + \int_c^d b(t,\xi)\Delta U(x,\sigma(t,\xi))d\eta(\xi) + F(x,t), \quad for \;\; t \neq t_k \\ U(x,t_k^+) = \alpha_k(x,t_k,U(x,t_k)), \\ \frac{\partial^\alpha}{\partial t^\alpha}U(x,t_k^+) = \beta_k\left(x,t_k,\frac{\partial^\alpha}{\partial t^\alpha}U(x,t_k)\right), \quad \text{for} \quad t = t_k,\; k = 1,2,\cdots,\; and \;\; (x,t) \in \Omega \times \mathbb{R}_+ \equiv G, \end{cases} \tag{E}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a piecewise smooth boundary $\partial\Omega$ and Δ is the Laplacian in the Euclidean space $\mathbb{R}^N$, $\frac{\partial^\alpha}{\partial t^\alpha}$ denotes the conformable partial fractional derivative of order α, $0 < \alpha \leq 1$, and $\mathbb{R}_+ = [0,+\infty)$. Moreover, we consider the following boundary condition:

$$U(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}_+. \tag{B_1}$$

Next, we present the following set of conditions which we assume to hold, throughout the paper.

$(\boldsymbol{H_1})$ $r(t) \in C^\alpha(\mathbb{R}_+,(0,+\infty))$, $b(t,\xi) \in C(\mathbb{R}_+ \times [a,b],\mathbb{R}_+)$, $a(t) \in PC(\mathbb{R}_+,\mathbb{R}_+)$, where PC denotes the class of functions which are piecewise continuous in t with discontinuities of the first kind only at $t = t_k$, $k = 1,2,\cdots$, and left continuous at $t = t_k$, $k = 1,2,\cdots$.

$(\boldsymbol{H_2})$ $q(x,t,\xi) \in C(\overline{\Omega} \times \mathbb{R}_+ \times [a,b],\mathbb{R}_+)$, $Q(t,\xi) = \min_{x\in\overline{\Omega}} q(x,t,\xi)$, $\theta(t,\xi)$, $\sigma(t,\xi) \in C(\mathbb{R}_+ \times [a,b],\mathbb{R})$, $\theta(t,\xi) \leq t$, $\sigma(t,\xi) \leq t$ for $\xi \in [a,b]$, $\theta(t,\xi)$ and $\sigma(t,\xi)$ are non-decreasing with respect to t and ξ respectively and

$$\liminf_{t\to+\infty,\xi\in[a,b]} \theta(t,\xi) = \liminf_{t\to+\infty,\xi\in[a,b]} \sigma(t,\xi) = +\infty.$$

There exists a function $\tau(t) \in C^\alpha(\mathbb{R}_+,\mathbb{R}_+)$ satisfying $\tau(t) \leq \theta(t,c)$, with $\tau'(t) > 0$ and $\lim_{t\to+\infty}\tau(t) = +\infty$.

$(\boldsymbol{H_3})$ $\eta(\xi)\colon [a,b] \to \mathbb{R}$ is nondecreasing and the integral is a stieltjes integral in (E), $F \in C(\bar{G},\mathbb{R}^M)$, $f_H(x,t) \in C(\bar{G},\mathbb{R})$ and $\int_\Omega f_H(x,t)dx \leq 0$, $f \in C(\mathbb{R},\mathbb{R})$ are convex and nondecreasing in $\mathbb{R}_+$ with $uf(u) > 0$ for $u \neq 0$ and there exists a positive constant ε such that $\frac{f(u)}{u} \geq \varepsilon > 0$ for $u \neq 0$.

$(\boldsymbol{H_4})$ All the components of $U(x,t)$ and their derivative $\frac{\partial^\alpha}{\partial t^\alpha}U(x,t)$ are piecewise continuous in t with discontinuities of the first kind only at $t = t_k$, $k = 1,2,\cdots$, and left continuous at $t = t_k$

$$U(x,t_k) = U(x,t_k^-), \quad \frac{\partial^\alpha}{\partial t^\alpha}U(x,t_k) = \frac{\partial^\alpha}{\partial t^\alpha}U(x,t_k^-), \quad k = 1,2,\cdots.$$

$(\boldsymbol{H_5})$ $\alpha_k,\beta_k \in PC(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R},\mathbb{R})$ for $k = 1,2,\cdots$, and there exist constants a_k, a_k^*, b_k, b_k^* such that for $k = 1,2,\cdots$,

$$a_k^* \leq \frac{\alpha_k(x,t_k,U(x,t_k))}{U(x,t_k)} \leq a_k, \qquad b_k^* \leq \frac{\beta_k\left(x,t_k,\frac{\partial^\alpha}{\partial t^\alpha}U(x,t_k)\right)}{\frac{\partial^\alpha}{\partial t^\alpha}U(x,t_k)} \leq b_k.$$

In Section 2, we present the definitions and introduce the notation we will use through the paper. In Section 3, we discuss the H-oscillation of the boundary problem $(E)-(B_1)$. In Section 4 we present some examples to illustrated the main result.

2. Preliminaries

In this section, we present some definitions and review some noteworthy results, from the literature which we will use throughout the paper.

Definition 2.1 [28]. By a **solution** of $(\mathrm{E})-(\mathrm{B}_1)$ we mean a function $\mathrm{U}(\mathrm{x},\mathrm{t}) \in \mathrm{C}^{2\alpha}(\overline{\Omega} \times [\mathrm{t}_1,+\infty),\mathbb{R}^\mathrm{M}) \cap \mathrm{C}^\alpha(\overline{\Omega} \times [\hat{\mathrm{t}}_1,+\infty),\mathbb{R}^\mathrm{M})$ which satisfies (E), where

$$\mathrm{t}_1 := \min\left\{0, \min_{\xi\in[c,d]}\left\{\inf_{t\geq 0}\sigma(t,\xi)\right\}\right\} \quad \text{and} \quad \hat{\mathrm{t}}_1 := \min\left\{0, \min_{\xi\in[c,d]}\left\{\inf_{t\geq 0}\theta(t,\xi)\right\}\right\}.$$

Now based on this definition of a solution, we can precisely define what we mean by *H-oscillation*.

Definition 2.2 *[28]*. Let H be a fixed unit vector in $\mathbb{R}^M$. A solution $U(x,t)$ of (E) is said to be **H-oscillatory** in G if the inner product $\langle U(x,t),H\rangle$ has a zero in $\Omega\times[t,+\infty)$ for $t>0$. Otherwise it is a H-nonoscillatory.

We use the following definition introduced by Khalil et al. in [11].

Definition 2.3 Let $f:[0,\infty)\to\mathbb{R}$. Then the conformable fractional derivative • of f of order α is defined by

$$T_\alpha(f)(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}$$

for all $t>0,\alpha\in(0,1]$.

If f is α-differentiable in some $(0,a),a>0$ and $\lim_{t\to 0^+}f^{(\alpha)}(t)$ exists, then we define

$$f^{(\alpha)}(0)=\lim_{t\to 0^+}f^{(\alpha)}(t).$$

Definition 2.4 $I_\alpha^a(f)(t)=I_1^a(t^{\alpha-1}f)=\int_a^t\frac{f(x)}{x^{1-\alpha}}dx$, where the integral is the usual Riemann improper integral, and $\alpha\in(0,1)$.

The following theorem defines the fundamental properties of the conformable fractional derivative:

Theorem 2.5. Let $\alpha\in(0,1]$ and f,g be α- differentiable at a point $t>0$. Then

(i) $T_\alpha(af+bg)=aT_\alpha(f)+bT_\alpha(g)$, for all $a,b\in\mathbb{R}$.

(ii) $T_\alpha(t^p)=pt^{p-\alpha}$, for all $p\in\mathbb{R}$.

(iii) $T_\alpha(\lambda)=0$, for all constant functions $f(t)=\lambda$.

(iv) $T_\alpha(fg)=fT_\alpha(g)+gT_\alpha(f)$. $T_\alpha\left(\frac{f}{g}\right)=\frac{gT_\alpha(f)-fT_\alpha(g)}{g^2}$.

(v) If f is differentiable, then $T_\alpha(f)(t)=t^{1-\alpha}\frac{df}{dt}(t)$.

Definition 2.6 [1]. Let f be a function of n variables $x_1,x_2,\cdots,x_n$ and the conformable partial derivative of f of order $0<\alpha\le 1$ in x_i is defined as follows:

$$\frac{\partial^\alpha}{\partial x_i^\alpha}f(x_1,x_2,\cdots,x_n)=\lim_{\varepsilon\to 0}\frac{f(x_1,x_2,\cdots,x_{i-1},x_i+\varepsilon x_i^{1-\alpha},\cdots,x_n)-f(x_1,x_2,\cdots,x_n)}{\varepsilon}$$

Next, we state two results which will help us establish our main results.

Lemma 2.7 [9]. If X and Y are nonnegative, then

$$X^\delta-\delta XY^{\delta-1}+(\delta-1)Y^\delta\ge 0,\quad \text{if}\quad \delta>1$$
$$X^\delta-\delta XY^{\delta-1}-(1-\delta)Y^\delta\le 0,\quad \text{if}\quad 0<\delta<1.$$

In both cases, equality holds, if and only if $X=Y$.

It is known [26] that the first eigenvalue λ_0 of the problem

$$\begin{cases}\Delta w(x)+\lambda w(x) & =0 \quad \text{in} \quad \Omega\\ w(x) & =0 \quad \text{on} \quad \partial\Omega\end{cases}$$

is positive and the corresponding eigenfunction $\Phi(x)$ is positive in Ω.

For convenience, we introduce the following notation:

$$u_H(x,t)=\langle U(x,t),H\rangle,\quad G(t)=\int_c^d\varepsilon Q(t,\xi)d\eta(\xi),$$
$$V_H(t)=K_\Phi\int_\Omega u_H(x,t)\Phi(x)dx,\quad f_H(x,t)=\langle F(x,t),H\rangle$$

where $K_\Phi=\left(\int_\Omega\Phi(x)dx\right)^{-1}$.

3. The H-Oscillations of the Problem $(E)-(B_1)$

In this section, we establish sufficient conditions for the H-oscillation of all solutions of the boundary problem $(E)-(B_1)$.

Lemma 3.1. Let H be a fixed unit vector in $\mathbb{R}^M$ and $U(x,t)$ be a solution of (E).

1. If $u_H(x,t)$ is eventually positive, then $u_H(x,t)$ satisfies the scalar impulsive partial conformable fractional differential inequality

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$$\begin{cases}\frac{\partial^\alpha}{\partial t^\alpha}\left(r(t)\frac{\partial^\alpha}{\partial t^\alpha}(u_H(x,t))\right)+\int_c^d \varepsilon Q(t,\xi)u_H(x,\theta(t,\xi))d\eta(\xi)\\ -a(t)\Delta u_H(x,t)-\int_c^d b(t,\xi)\Delta u_H(x,\sigma(t,\xi))d\eta(\xi)\le f_H(x,t),\quad t\neq t_k \\ a_k^*\le\frac{u_H(x,t_k^+)}{u_H(x,t_k)}\le a_k,\quad b_k^*\le\frac{\frac{\partial^\alpha}{\partial t^\alpha}u_H(x,t_k^+)}{\frac{\partial^\alpha}{\partial t^\alpha}u_H(x,t_k)}\le b_k,\quad k=1,2,\cdots.\end{cases}\tag{1}$$

2. If $u_H(x,t)$ is eventually negative, then $u_H(x,t)$ satisfies the scalar impulsive partial conformable fractional differential inequality

$$\begin{cases}\frac{\partial^\alpha}{\partial t^\alpha}\left(r(t)\frac{\partial^\alpha}{\partial t^\alpha}(u_H(x,t))\right)+\int_c^d \varepsilon Q(t,\xi)u_H(x,\theta(t,\xi))d\eta(\xi)\\ -a(t)\Delta u_H(x,t)-\int_c^d b(t,\xi)\Delta u_H(x,\sigma(t,\xi))d\eta(\xi)\ge f_H(x,t),\quad t\neq t_k \\ a_k^*\ge\frac{u_H(x,t_k^+)}{u_H(x,t_k)}\ge a_k,\quad b_k^*\ge\frac{\frac{\partial^\alpha}{\partial t^\alpha}u_H(x,t_k^+)}{\frac{\partial^\alpha}{\partial t^\alpha}u_H(x,t_k)}\ge b_k,\quad k=1,2,\cdots.\end{cases}\tag{2}$$

Proof. (i) Let $u_H(x,t)$ be eventually positive.
Case:(i) $t\neq t_k$, $k=1,2,\cdots$. Taking the inner product of (E) and H, we get

$$\frac{\partial^\alpha}{\partial t^\alpha}\left(r(t)\frac{\partial^\alpha}{\partial t^\alpha}(\langle U(x,t),H\rangle)\right)+\int_c^d q(x,t,\xi)f(\langle U(x,\theta(t,\xi)),H\rangle)d\eta(\xi)$$
$$=a(t)\Delta\langle U(x,t),H\rangle+\int_c^d b(t,\xi)\Delta\langle U(x,\sigma(t,\xi)),H\rangle d\eta(\xi)+\langle F(x,t),H\rangle,$$

that is

$$\begin{cases}\frac{\partial^\alpha}{\partial t^\alpha}\left(r(t)\frac{\partial^\alpha}{\partial t^\alpha}(u_H(x,t))\right)+\int_c^d q(x,t,\xi)f\left(u_H(x,\theta(t,\xi))\right)d\eta(\xi)\\ =a(t)\Delta u_H(x,t)+\int_c^d b(t,\xi)\Delta u_H(x,\sigma(t,\xi))d\eta(\xi)+f_H(x,t).\end{cases}\tag{3}$$

Using condition (H_2) and Jensen's inequality, we have

$$\int_c^d q(x,t,\xi)f\left(u_H(x,\theta(t,\xi))\right)d\eta(\xi)\ge\varepsilon\int_c^d Q(t,\xi)u_H(x,\theta(t,\xi))d\eta(\xi).\tag{4}$$

From (3) and (4), it follows that

$$\begin{cases}\frac{\partial^\alpha}{\partial t^\alpha}\left(r(t)\frac{\partial^\alpha}{\partial t^\alpha}(u_H(x,t))\right)+\int_c^d \varepsilon Q(t,\xi)u_H(x,\theta(t,\xi))d\eta(\xi)\\ -a(t)\Delta u_H(x,t)-\int_c^d b(t,\xi)\Delta u_H(x,\sigma(t,\xi))d\eta(\xi)\le f_H(x,t),\quad t\neq t_k.\end{cases}\tag{5}$$

Case:(ii) $t=t_k$, $k=1,2,\cdots$. Taking the inner product of (E) and H, and using (H_5), we obtain

$$a_k^*\le\frac{U(x,t_k^+)}{U(x,t_k)}\le a_k,\quad b_k^*\le\frac{\frac{\partial^\alpha}{\partial t^\alpha}U(x,t_k^+)}{\frac{\partial^\alpha}{\partial t^\alpha}U(x,t_k)}\le b_k$$

$$a_k^*\le\frac{\langle U(x,t_k^+),H\rangle}{\langle U(x,t_k),H\rangle}\le a_k,\quad b_k^*\le\frac{\frac{\partial^\alpha}{\partial t^\alpha}\langle U(x,t_k^+),H\rangle}{\frac{\partial^\alpha}{\partial t^\alpha}\langle U(x,t_k),H\rangle}\le b_k$$

that is

$$a_k^*\le\frac{u_H(x,t_k^+)}{u_H(x,t_k)}\le a_k,\quad b_k^*\le\frac{\frac{\partial^\alpha}{\partial t^\alpha}u_H(x,t_k^+)}{\frac{\partial^\alpha}{\partial t^\alpha}u_H(x,t_k)}\le b_k,\quad k=1,2,\cdots.\tag{6}$$

Therefore, combining (5) and (6) we immediately obtain (1), which shows that $u_H(x,t)$ satisfies the scalar impulsive partial conformable differential inequality (1).

(ii) The proof is similar to that of (i) and we omit it. The proof is complete. □

Let H be a fixed unit vector in $\mathbb{R}^M$. The inner product of boundary conditions (B_1) and H yields the following boundary conditions:

$$u_H(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}_+. \qquad (B_{11})$$

Lemma 3.2. Let H be a fixed unit vector in $\mathbb{R}^M$. If the scalar impulsive partial conformable fractional differential inequality (1) has no eventually positive solutions and the scalar impulsive partial conformable fractional differential inequality (2) has no eventually negative solutions, satisfying the boundary conditions (B_{11}), then every solution $U(x,t)$ of the problem $(E)-(B_1)$ is H-oscillatory in G.

Proof. Suppose to the contrary that there is a H-nonoscillatory solution $U(x,t)$ of the problem $(E)-(B_1)$ in G, then $u_H(x,t)$ is eventually positive or eventually negative. If $u_H(x,t)$ is eventually positive, by Lemma 3.1, easily, we obtain that $u_H(x,t)$ satisfies the scalar impulsive partial conformable fractional differential inequality (1). On the other hand, it is easy to see that $u_H(x,t)$ satisfies the boundary conditions (B_{11}). This is a contradiction to the hypothesis.

Similarly, if $u_H(x,t)$ is eventually negative using Lemma 3.1, easily, we obtain that $u_H(x,t)$ satisfies the scalar impulsive partial conformable fractional differential inequality (2). It is obvious that $u_H(x,t)$ satisfies, the boundary conditions (B_{11}). This is a contradiction. The proof is complete.

□

Theorem 3.3. Let H be a fixed unit vector in $\mathbb{R}^M$. If the impulsive conformable fractional differential inequality

$$\begin{cases} T_\alpha\left(r(t)T_\alpha\left(V_H(t)\right)\right) + G(t)V_H\left(\tau(t)\right) \le 0, \quad t \ne t_k \\ a_k^* \le \dfrac{V_H(t_k^+)}{V_H(t_k)} \le a_k, \quad b_k^* \le \dfrac{T_\alpha\left(V_H(t_k^+)\right)}{T_\alpha\left(V_H(t_k)\right)} \le b_k, \quad k = 1,2,\cdots, \end{cases} \qquad (7)$$

has no eventually positive solutions and the impulsive conformable fractional differential inequality

$$\begin{cases} T_\alpha\left(r(t)T_\alpha\left(V_H(t)\right)\right) + G(t)V_H\left(\tau(t)\right) \ge 0, \quad t \ne t_k \\ a_k^* \ge \frac{V_H(t_k^+)}{V_H(t_k)} \ge a_k, \quad b_k^* \ge \frac{T_\alpha\left(V_H(t_k^+)\right)}{T_\alpha\left(V_H(t_k)\right)} \ge b_k, \quad k = 1,2,\cdots, \end{cases} \qquad (8)$$

has no eventually negative solutions, then every solution $U(x,t)$ of the problem $(E)-(B_1)$ is H-oscillatory in G.

Proof. Suppose to the contrary that there exists a solution $U(x,t)$ of $(E)-(B_1)$ which is not H-oscillatory in G. Without loss of generality, we may assume that $u_H(x,t) > 0$ in $\Omega \times [t_0,+\infty)$, for some $t_0 > 0$. By the assumption that there exists a $t_1 > t_0$ such that $\theta(t,\xi) \le t_0$ and $\sigma(t,\xi) \le t_0$ for $(t,\xi) \in [t_1,+\infty) \times [a,b]$ we have that $u_H(x,\theta(t,\xi)) > 0$ and $u_H(x,\sigma(t,\xi)) > 0$, for $x \in \Omega$, $t \in [t_1,+\infty)$, $\xi \in [a,b]$.

For $t \ge t_0$ and $t \ne t_k$ for $k = 1,2,\cdots$, we multiply both sides of inequality (1) by $K_\Phi \Phi(x)$ and integrate with respect to x over the domain Ω to attain

$$\begin{cases} t^{1-\alpha}\dfrac{d}{dt}\left(r(t)t^{1-\alpha}\dfrac{d}{dt}\left(K_\Phi\displaystyle\int_\Omega u_H(x,t)\Phi(x)dx\right)\right) + \varepsilon K_\Phi \displaystyle\int_\Omega\int_c^d Q(t,\xi)u_H(x,\theta(t,\xi))\Phi(x)d\eta(\xi)dx \\ -a(t)K_\Phi\displaystyle\int_\Omega \Delta u_H(x,t)\Phi(x)dx - K_\Phi\displaystyle\int_\Omega\int_c^d b(t,\xi)\Delta u_H\big(x,\sigma(t,\xi)\big)\Phi(x)d\eta(\xi)dx \\ \le K_\Phi\displaystyle\int_\Omega f_H(x,t)\Phi(x)dx, \quad t \ne t_k. \end{cases} \qquad (9)$$

Using Green's formula and the boundary condition (B_{11}), we have that

$$K_\Phi\int_\Omega \Phi(x)\Delta u_H(x,t)dx = K_\Phi\int_{\partial\Omega}\left[\Phi(x)\frac{\partial u_H(x,t)}{\partial\gamma} - u_H(x,t)\frac{\partial\Phi(x)}{\partial\gamma}\right]dS + K_\Phi\int_\Omega u_H(x,t)\Delta\Phi(x)dx$$
$$= 0 - \lambda_0 V_H(t) \le 0, \qquad (10)$$

and

$$K_\Phi\int_\Omega\int_c^d b(t,\xi)\Phi(x)\Delta u_H(x,\sigma(t,\xi))d\eta(\xi)dx$$
$$= K_\Phi\int_c^d b(t,\xi)\int_{\partial\Omega}\left[\Phi(x)\frac{\partial u_H(x,\sigma(t,\xi))}{\partial\gamma} - u_H(x,\sigma(t,\xi))\frac{\partial\Phi(x)}{\partial\gamma}\right]dSd\eta(\xi)$$
$$+K_\Phi\int_c^d b(t,\xi)\int_\Omega u_H(x,\sigma(t,\xi))\Delta\Phi(x)dxd\eta(\xi)$$
$$= 0 - \lambda_0 V_H(\sigma(t,\xi)) \le 0 \qquad (11)$$

where dS is the surface element on $\partial\Omega$. Moreover, by (H_3), $\int_\Omega f_H(x,t)dx \le 0$. Combining (9)-(11) we get

$$t^{1-\alpha}\frac{d}{dt}\left[r(t)t^{1-\alpha}\frac{d}{dt}(V_H(t))\right] + \int_c^d \varepsilon Q(t,\xi)V_H(\theta(t,\xi))d\eta(\xi) \le 0, \quad t \ge t_0.$$

From (H_2) and $T_\alpha(V_H(t)) > 0$, we have

$$V_H(\theta(t,\xi)) \geq V_H(\theta(t,c)) > 0, \quad \xi \in [c,d] \quad and \quad \tau(t) \leq \theta(t,c) \leq t.$$

Thus, $V_H(\tau(t)) \leq V_H(\theta(t,c))$ and therefore

$$T_\alpha(r(t)T_\alpha(V_H(t))) + G(t)V_H(\tau(t)) \leq 0, \quad t \geq t_1. \tag{12}$$

For $t \geq t_0$, $t = t_k$, $k = 1,2,\cdots$, multiplying both sides of inequality (1) by $K_\Phi \Phi(x)$ and integrating with respect to x over the domain Ω, we obtain

$$a_k^* \leq \frac{V_H(t_k^+)}{V_H(t_k)} \leq a_k, \quad b_k^* \leq \frac{T_\alpha(V_H(t_k^+))}{T_\alpha(V_H(t_k))} \leq b_k, \quad k = 1,2,\cdots. \tag{13}$$

Therefore (12) and (13) show that $V_H(t) > 0$ is a positive solution of the impulsive conformable fractional differential inequality (7). This is a contradiction.

Suppose, now, that $u_H(x,t) < 0$ is a negative solution of the impulsive partial conformable fractional differential inequality (2) satisfying the boundary condition (B_1), $(x,t) \in \Omega \times [t_0,+\infty)$, $t_0 > 0$. Using the above procedure, easily, we can reach a contradiction. The proof is complete. □

Theorem 3.4. Assume that $(H_1) - (H_5)$ hold, and

$$\int_{t_0}^{+\infty} \frac{s^{\alpha-1}}{r(s)} ds = +\infty. \tag{14}$$

Moreover, if there exists a function $\rho(t) \in C^\alpha(\mathbb{R}_+,(0,+\infty))$ which is nondecreasing with respect to t, such that

$$\limsup_{t\to+\infty} \int_{t_0}^{t} \prod_{t_0 \leq t_k < s} \left(\frac{b_k}{a_k^*}\right)^{-1} s^{\alpha-1}\left[\rho(s)G(s) - \frac{E^2(s)}{4F(s)}\right] ds = +\infty, \tag{15}$$

where

$$E(t) = t^{1-\alpha}\frac{\rho'(t)}{\rho(t)} \quad \text{and} \quad F(t) = \frac{t^{1-\alpha}\tau'(t)}{\rho(\tau(t))r(\tau(t))},$$

then every solution of $U(x,t)$ of the boundary value problem $(E) - (B_1)$ is H-oscillatory in G.

Proof. We prove that the inequality (7) has no eventually positive solution if the conditions of Theorem 3.3 hold. Suppose that $V_H(t)$ is an eventually positive solution of the inequality (7) then there exists a number $t_1 \geq t_0$ such that $V_H(\tau(t)) > 0$ for $t \geq t_1$.

Thus we have

$$T_\alpha(r(t)T_\alpha(V_H(t))) + G(t)V_H(\tau(t)) \leq 0. \tag{16}$$

Thus $T_\alpha[r(t)T_\alpha(V_H(t))] \geq 0$ or $T_\alpha[r(t)T_\alpha(V_H(t))] < 0$, $t \geq t_1$ for some $t_1 \geq t_0$. We now claim that

$$T_\alpha[r(t)T_\alpha(V_H(t))] \geq 0 \quad for \quad t \geq t_1. \tag{17}$$

Suppose this is false. Then $T_\alpha[r(t)T_\alpha(V_H(t))] < 0$ and there exists $t_2 \in [t_1,+\infty)$ such that $T_\alpha[r(t_2)T_\alpha(V_H(t_2))] < 0$. Since $r(t)T_\alpha(V_H(t))$ is strictly decreasing on $[t_1,+\infty)$, it is clear that

$$r(t)T_\alpha(V_H(t)) < r(t_2)T_\alpha(V_H(t_2)) := -\mu$$

where $\mu > 0$ is a constant. For $t \in [t_2,+\infty)$, we have

$$T_\alpha(V_H(t)) < -\frac{\mu}{r(t)}.$$

Integrating the above inequality from t_2 to t, we have

$$V_H(t) \leq V_H(t_2) - \mu\int_{t_2}^{t} s^{\alpha-1}\left(\frac{1}{r(s)}\right) ds.$$

Letting $t \to +\infty$, we get $\lim_{t\to+\infty} V_H(t) = -\infty$ which is contradiction. That proves that (17) holds. We define the Riccati Transformation

$$W(t) := \rho(t)\frac{r(t)T_\alpha(V_H(t))}{V_H(\tau(t))}.$$

Then $W(t) \geq 0$ and

$$T_\alpha(W(t)) \leq t^{1-\alpha}\frac{\rho'(t)}{\rho(t)}W(t) - \rho(t)G(t) - \frac{W^2(t)}{\rho(\tau(t))}\frac{\tau'(t)}{r(\tau(t))}t^{1-\alpha}.$$

Thus $T_\alpha(W(t)) \leq E(t)W(t) - G(t)\rho(t) - W^2(t)F(t)$ and $W(t_k^+) \leq \frac{b_k}{a_k^*}W(t_k)$. Define

$$A(t) = \prod_{t_0 \leq t_k < t}\left(\frac{b_k}{a_k^*}\right)^{-1} W(t).$$

It is clear that $W(t)$ is continuous in each interval $(t_k,t_{k+1}]$, and since $W(t_k^+) \leq \frac{b_k}{a_k^*}W(t_k)$, it follows that

$$A(t_k^+) = \prod_{t_0 \leq t_j \leq t_k}\left(\frac{b_k}{a_k^*}\right)^{-1} W(t_k^+) \leq \prod_{t_0 \leq t_j < t_k}\left(\frac{b_k}{a_k^*}\right)^{-1} W(t_k) = A(t_k).$$

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For all $t \geq t_0$,

$$A(t_k^-) = \prod_{t_0 \leq t_j \leq t_{k-1}} \left(\frac{b_k}{a_k^*}\right)^{-1} W(t_k^-) \leq \prod_{t_0 \leq t_j < t_k} \left(\frac{b_k}{a_k^*}\right)^{-1} W(t_k) = A(t_k)$$

which implies that $A(t)$ is continuous on $[t_0, +\infty)$.

$$T_\alpha(A(t)) + \prod_{t_0 \leq t_k < t} \left(\frac{b_k}{a_k^*}\right) A^2(t)F(t) + \prod_{t_0 \leq t_k < t} \left(\frac{b_k}{a_k^*}\right)^{-1} G(t)\rho(t) - A(t)E(t)$$
$$= \prod_{t_0 \leq t_k < t} \left(\frac{b_k}{a_k^*}\right)^{-1} [T_\alpha(W(t)) + W^2(t)F(t) - W(t)E(t) + G(t)\rho(t)] \leq 0,$$

that is

$$T_\alpha(A(t)) \leq -\prod_{t_0 \leq t_k < t} \left(\frac{b_k}{a_k^*}\right) F(t)A^2(t) + A(t)E(t) - \prod_{t_0 \leq t_k < t} \left(\frac{b_k}{a_k^*}\right)^{-1} G(t)\rho(t).$$

Taking

$$X(t) = \left(\prod_{t_0 \leq t_k < t} \left(\frac{b_k}{a_k^*}\right) F(t)\right)^{\frac{1}{2}} A(t) \quad \text{and} \quad Y(t) = \frac{E(t)}{2}\left(\prod_{t_0 \leq t_k < t} \left(\frac{b_k}{a_k^*}\right)^{-1} \frac{1}{F(t)}\right)^{\frac{1}{2}},$$

and using Lemma 2.7, we have

$$E(t)A(t) - \prod_{t_0 \leq t_k < t} \left(\frac{b_k}{a_k^*}\right) F(t)A^2(t) \leq \frac{E^2(t)}{4F(t)} \prod_{t_0 \leq t_k < t} \left(\frac{b_k}{a_k^*}\right)^{-1}.$$

Thus

$$T_\alpha(A(t)) \leq -\prod_{t_0 \leq t_k < t} \left(\frac{b_k}{a_k^*}\right)^{-1} \left[G(t)\rho(t) - \frac{E^2(t)}{4F(t)}\right].$$

Integrating both sides from t_0 to t, we have

$$A(t) \leq A(t_0) - \int_{t_0}^{t} \prod_{t_0 \leq t_k < s} \left(\frac{b_k}{a_k^*}\right)^{-1} s^{\alpha-1} \left[G(s)\rho(s) - \frac{E^2(s)}{4F(s)}\right] ds.$$

Thus

$$\limsup_{t \to +\infty} \int_{t_0}^{t} \prod_{t_0 \leq t_k < s} \left(\frac{b_k}{a_k^*}\right)^{-1} s^{\alpha-1} \left[G(s)\rho(s) - \frac{E^2(s)}{4F(s)}\right] ds < \infty.$$

which contradicts (15). The proof of the theorem is complete. □

Theorem 3.5. Suppose that $(H_1) - (H_5)$ and (17) hold. Furthermore assume that there exist functions ρ and $\phi \in C^\alpha(\mathbb{R}_+, (0, +\infty))$ in which ρ is nondecreasing and functions $b, B \in C(\mathbb{D}, \mathbb{R})$, where $\mathbb{D} = \{(t,s): t \geq s \geq t_0 > 0\}$ such that

$(\boldsymbol{H_6})$ $B(t,t) = 0$ and $B(t,s) > 0$ for all $t > s \geq t_0$,

$(\boldsymbol{H_7})$ $\frac{\partial B(t,s)}{\partial t} \geq 0$ and $\frac{\partial B(t,s)}{\partial s} \leq 0$,

$(\boldsymbol{H_8})$ $-\frac{\partial B(t,s)}{\partial s} = b(t,s)\sqrt{B(t,s)}$.

If

$$\limsup_{t \to +\infty} \frac{1}{B(t,t_0)} \int_{t_0}^{t} \prod_{t_0 \leq t_k < s} \left(\frac{b_k}{a_k^*}\right)^{-1} \Pi(s) ds = +\infty, \tag{18}$$

where

$$\Pi(s) = G(s)\rho(s)B(t,s)\phi(s) - \frac{1}{4}\frac{[b(t,s)\phi(s)s^{1-\alpha} - \phi'(s)\sqrt{B(t,s)}s^{1-\alpha} - E(s)\phi(s)\sqrt{B(t,s)} - (1-\alpha)s^{-\alpha}\phi(s)\sqrt{B(t,s)}]^2}{F(s)\phi(s)}$$

then every solution of $U(x,t)$ of the boundary value problem $(E) - (B_1)$ is H-oscillatory in G.

Proof. Suppose that $U(x,t)$ is a H-nonoscillatory solution of the boundary value problem $(E) - (B_1)$. Without loss of generality we can assume that $u_H(x,t)$ be an eventually positive solution. Let $V_H(t)$ is an eventually positive solution of (7). Proceeding as in the proof of Theorem 3.4 we obtain

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$$T_\alpha(A(t)) \le -\prod_{t_0\le t_k<t}\left(\frac{b_k}{a_k^*}\right)F(t)A^2(t)+A(t)E(t)-\prod_{t_0\le t_k<t}\left(\frac{b_k}{a_k^*}\right)^{-1}G(t)\rho(t).$$

Multiplying the above inequality by $B(t,s)\phi(s)$ for $t\ge s\ge T$, and integrating from T to t, we have

$$\int_T^t s^{1-\alpha}A'(s)B(t,s)\phi(s)ds \le -\int_T^t\prod_{t_0\le t_k<s}\left(\frac{b_k}{a_k^*}\right)F(s)A^2(s)B(t,s)\phi(s)ds$$
$$+\int_T^t A(s)E(s)B(t,s)\phi(s)ds-\int_T^t\prod_{t_0\le t_k<s}\left(\frac{b_k}{a_k^*}\right)^{-1}G(s)\rho(s)B(t,s)\phi(s)ds.$$

Thus

$$\int_T^t\prod_{t_0\le t_k<s}\left(\frac{b_k}{a_k^*}\right)^{-1}G(s)\rho(s)B(t,s)\phi(s)ds$$
$$\le A(T)B(t,T)\phi(T)T^{1-\alpha}$$
$$-\int_T^t\left[-\frac{\partial B(t,s)}{\partial s}\phi(s)s^{1-\alpha}-B(t,s)\phi'(s)s^{1-\alpha}-(1-\alpha)B(t,s)\phi(s)s^{-\alpha}-E(s)B(t,s)\phi(s)\right]A(s)ds$$
$$-\int_T^t\prod_{t_0\le t_k<s}\left(\frac{b_k}{a_k^*}\right)F(s)A^2(s)B(t,s)\phi(s)ds.$$

Therefore

$$\int_T^t\prod_{t_0\le t_k<s}\left(\frac{b_k}{a_k^*}\right)^{-1}G(s)\rho(s)B(t,s)\phi(s)ds$$
$$-\frac{1}{4}\int_T^t\prod_{t_0\le t_k<s}\left(\frac{b_k}{a_k^*}\right)^{-1}$$
$$\times\frac{\left[b(t,s)\phi(s)s^{1-\alpha}-\phi'(s)\sqrt{B(t,s)}s^{1-\alpha}-E(s)\phi(s)\sqrt{B(t,s)}-(1-\alpha)s^{-\alpha}\phi(s)\sqrt{B(t,s)}\right]^2}{F(s)\phi(s)}ds$$
$$\le A(T)B(t,T)\phi(T)T^{1-\alpha}. \qquad (19)$$

From (19) for $t\ge T\ge t_0$, we have

$$\frac{1}{B(t,t_0)}\int_{t_0}^t\prod_{t_0\le t_k<s}\left(\frac{b_k}{a_k^*}\right)^{-1}\times$$
$$\left[G(s)\rho(s)B(t,s)\phi(s)\right.$$
$$\left.-\frac{1}{4}\frac{\left[b(t,s)\phi(s)s^{1-\alpha}-\phi'(s)\sqrt{B(t,s)}s^{1-\alpha}-E(s)\phi(s)\sqrt{B(t,s)}-(1-\alpha)s^{-\alpha}\phi(s)\sqrt{B(t,s)}\right]^2}{F(s)\phi(s)}\right]ds$$
$$\le\int_{t_0}^T\prod_{t_0\le t_k<s}\left(\frac{b_k}{a_k^*}\right)^{-1}G(s)\rho(s)\phi(s)ds+\phi(T)A(T)T^{1-\alpha}.$$

Letting $t\to+\infty$, we have

$$\limsup_{t\to+\infty}\frac{1}{B(t,t_0)}\int_{t_0}^t\prod_{t_0\le t_k<s}\left(\frac{b_k}{a_k^*}\right)^{-1}\times$$
$$\left[G(s)\rho(s)B(t,s)\phi(s)\right.$$
$$\left.-\frac{1}{4}\frac{\left[b(t,s)\phi(s)s^{1-\alpha}-\phi'(s)\sqrt{B(t,s)}s^{1-\alpha}-E(s)\phi(s)\sqrt{B(t,s)}-(1-\alpha)s^{-\alpha}\phi(s)\sqrt{B(t,s)}\right]^2}{F(s)\phi(s)}\right]ds$$
$$\le\int_{t_0}^T\prod_{t_0\le t_k<s}\left(\frac{b_k}{a_k^*}\right)^{-1}G(s)\rho(s)\phi(s)ds+\phi(T)A(T)T^{1-\alpha}<+\infty,$$

which contradicts (18). The proof of the theorem is complete. □

Choosing $\phi(s) = \rho(s) \equiv 1$, in Theorem 3.5, we establish the following result.

Corollary 3.6. Assume that the conditions of Theorem 3.5 hold and

$$\limsup_{t\to+\infty} \frac{1}{B(t,t_0)} \int_{t_0}^{t} \prod_{t_0 \le t_k < s} \left(\frac{b_k}{a_k^*}\right)^{-1} \times \left[G(s)B(t,s) - \frac{1}{4}\frac{\left[b(t,s)s^{1-\alpha} - E(s)\sqrt{B(t,s)} - (1-\alpha)s^{-\alpha}\sqrt{B(t,s)}\right]^2}{F(s)}\right] ds = +\infty.$$

Then every solution of $U(x,t)$ of the boundary value problem $(E)-(B_1)$ is H-oscillatory in G.

Next we consider the case

$$\int_{t_0}^{+\infty} s^{\alpha-1}\frac{1}{r(s)}ds < +\infty, \tag{20}$$

which implies that (14) does not hold. In this case, we get the following result.

Theorem 3.7. Assume that $(H_1)-(H_5)$ and (20) hold. Then every solution of $U(x,t)$ of the problem $(E)-(B_1)$ is H −oscillatory if

$$\int_{t_0}^{+\infty} \frac{s^{\alpha-1}}{r(s)} G(s)V_H(\theta(t))ds = +\infty. \tag{21}$$

Proof. Suppose that $U(x,t)$ is H-nonoscillatory of $(E)-(B_1)$. Without loss of generality, we can assume that $u_H(x,t)$ is an eventually positive solution. Then $V_H(t)$ is an eventually positive solution of (7). Proceeding as in the proof of Theorem 3.3, we consider two cases according to the sign of $T_\alpha(V_H(t))$. The proof when $T_\alpha(V_H(t))$ is eventually positive is similar to that of Theorem 3.3 and hence is omitted. Next assume that $T_\alpha(V_H(t))$ is eventually negative. Then there exists $t_3 \ge t_2$ such that $T_\alpha(V_H(t)) < 0$ for $t \ge t_3$. From (7) we have

$$T_\alpha(r(t)T_\alpha(V_H(t))) + G(t)V_H(\theta(t)) \le 0, \quad t \ge t_3.$$

Integrating from T to t, we have

$$\int_T^t \frac{s^{\alpha-1}}{r(s)} G(s)V_H(\theta(s))ds \le T_\alpha(V_H(T)), \quad t \ge T.$$

Therefore, letting $t \to +\infty$, we obtain

$$\int_T^{+\infty} \frac{s^{\alpha-1}}{r(s)} G(s)V_H(\theta(s))ds < \infty,$$

which contradicts (21). The proof is complete. □

4. Examples

In this section we provide two examples to illustrate main results.

Example 4.1. Consider the following impulsive partial conformable fractional differential equations

$$\begin{cases} \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}\left[4\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}(U(x,t))\right] + \frac{t}{2}\int_{\pi/2}^{\pi} U(x,t-\xi)d\xi \\ = 5t\Delta U(x,t) + \frac{3t}{2}\int_{\pi/2}^{\pi} \Delta U(x,t-\xi)d\xi + F(x,t), \quad t \ne t_k, \\ U(x,t_k^+) = \frac{k+1}{k}U(x,t_k), \\ \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}U(x,t_k^+) = \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}U(x,t_k), \quad k = 1,2,\cdots, \end{cases} \tag{22}$$

for $(x,t) \in (0,\pi) \times \mathbb{R}_+$, with the boundary condition

$$U(0,t) = U(\pi,t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad t \ne t_k, \ k = 1,2,\cdots. \tag{23}$$

Here $\Omega = (0,\pi)$, $N = 1$, $M = 2$, $\alpha = \frac{1}{2}$, $a_k = a_k^* = \frac{k+1}{k}$, $b_k = b_k^* = 1$, $r(t) = 4$, $\theta(t,\xi) = \sigma(t,\xi) = t - \xi$, $\eta(\xi) = \xi$, $Q(t,\xi) = \frac{t}{2}$, $a(t) = 5t$, $b(t,\xi) = \frac{3t}{2}$, $[a,b] = [\pi/2,\pi]$, $f(u) = u$, $\varepsilon = 1$,

$$F(x,t) = \begin{pmatrix} \left(2 - \frac{4t}{2}\right)\sin x \ \cos t \\ \sin x \ e^t\left(2 + 9t - \frac{4t}{2}e^{-\pi} + \frac{4t}{2}e^{-\pi/2}\right) \end{pmatrix}.$$

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Let $H = e_1 = \binom{1}{0}$, then we have $f_H(x,t) = f_{e_1}(x,t) = \left(2 - \frac{4t}{2}\right) sinxcost$ and

$$\int_\Omega f_{e_1}(x,t)dx = \left(2 - \frac{4t}{2}\right)\mathrm{cost}\int_\Omega \mathrm{sinx}dx = 2\left(2 - \frac{4t}{2}\right)\mathrm{cost} \le 0, \quad \pi/2 \le t \le 3\pi/2.$$

Take $\tau(t) = \rho(t) = t$. Since $t_0 = 1$, $t_k = 2^k$, $E(s) = s^{-3/2}$, $G(s) = \frac{s\pi}{2}$ and $F(s) = \frac{s^{-3/2}}{4}$. Then, the conditions $(H_1) - (H_5)$ hold and moreover

$$\lim_{t\to+\infty}\int_{t_0}^{t}\prod_{t_0\le t_k<s}\frac{b_k^*}{a_k}ds = \int_1^{+\infty}\prod_{1<t_k<s}\frac{k}{k+1}ds$$
$$= \int_1^{t_1}\prod_{1<t_k<s}\frac{k}{k+1}ds + \int_{t_1^+}^{t_2}\prod_{1<t_k<s}\frac{k}{k+1}ds + \int_{t_2^+}^{t_3}\prod_{1<t_k<s}\frac{k}{k+1}ds + \cdots$$
$$= 1 + \frac{1}{2}\times 2 + \frac{1}{2}\times\frac{2}{3}\times 2^2 + \cdots$$
$$= \sum_{n=0}^{\infty}\frac{2^n}{n+1} = +\infty.$$

Thus

$$\limsup_{t\to+\infty}\int_1^t\prod_{1<t_k<s}\frac{k+1}{k}s^{-1/2}\left(\frac{\pi s^2}{2} - s^{-3/2}\right)ds = +\infty.$$

Therefore all the conditions of Theorem 3.4 are satisfied and hence every solution $U(x,t)$ of equation (22)-(23) is e_1-oscillatory in G.

In fact

$$U(x,t) = \begin{pmatrix}\mathrm{sinx}\ \mathrm{sint}\\ \mathrm{sinx}\ e^t\end{pmatrix},$$

is one such solution of the problem (22)-(23). Note that the above solution $U(x,t)$ is not e_2-oscillatory in G, where $e_2 = \binom{0}{1}$.

Example 4.2. Consider the following impulsive partial conformable fractional differential equations

$$\begin{cases}\frac{\partial^{\frac{1}{3}}}{\partial t^{\frac{1}{3}}}\left[2\frac{\partial^{\frac{1}{3}}}{\partial t^{\frac{1}{3}}}(U(x,t))\right] + t^{1/3}\int_0^\pi U(x,t-2\xi)d\xi \\ = 2t^{4/3}\Delta U(x,t) + \frac{4}{3}\int_0^\pi \Delta U(x,t-2\xi)d\xi + F(x,t), \quad t\neq t_k, \\ U(x,t_k^+) = U(x,t_k), \\ \frac{\partial^{\frac{1}{3}}}{\partial t^{\frac{1}{3}}}U(x,t_k^+) = \frac{k}{k+1}\frac{\partial^{\frac{1}{3}}}{\partial t^{\frac{1}{3}}}U(x,t_k), \quad k = 1,2,\cdots,\end{cases} \tag{24}$$

for $(x,t)\in(0,\pi)\times\mathbb{R}_+$, with the boundary condition

$$U(0,t) = U(\pi,t) = \binom{0}{0}, \quad t\neq t_k, k = 1,2,\cdots. \tag{25}$$

Here $\Omega = (0,\pi)$, $N = 1$, $M = 2$, $\alpha = \frac{1}{3}$, $a_k = a_k^* = 1$, $b_k = b_k^* = \frac{k+1}{k}$, $r(t) = 2$, $\theta(t,\xi) = \sigma(t,\xi) = t - 2\xi$, $\eta(\xi) = \xi$, $Q(t,\xi) = t^{1/3}$, $a(t) = 2t^{4/3}$, $b(t,\xi) = \frac{4}{3}$, $[a,b] = [0,\pi]$, $f(u) = u$, $\varepsilon = 1$,

$$F(x,t) = \begin{pmatrix} -\frac{11}{3}\mathrm{sinx}\ \mathrm{sint} \\ \mathrm{sinx}\ e^{-t}\left(2t^{4/3} - \frac{11}{6}t^{1/3} + \frac{1}{2}t^{1/3}e^{2\pi}\right)\end{pmatrix}.$$

Let $H = e_1 = \binom{1}{0}$, then we have $f_H(x,t) = f_{e_1}(x,t) = -\frac{11}{3}\mathrm{sinxsint}$ and

$$\int_\Omega f_{e_1}(x,t)dx = -\frac{11}{3}\mathrm{sint}\int_\Omega \mathrm{sinx}dx = -\frac{22}{3}2\mathrm{sint} \le 0, \quad 0 < t < \pi.$$

Take $\tau(t) = \rho(t) = t$. Since $t_0 = 1$, $t_k = 2^k$, $E(s) = s^{-1/3}$, $G(s) = s^{1/3}\pi$ and $F(s) = \frac{s^{-1/3}}{2}$. Thus

$$\limsup_{t\to+\infty}\int_1^t\prod_{1<t_k<s}\frac{k+1}{k}s^{-2/3}\left(s^{4/3}\pi - 2s^{-1/3}\right)ds = +\infty.$$

Therefore all the conditions of Theorem 3.4 are satisfied and hence every solution $U(x,t)$ of equation (24)-(25) is e_1-oscillatory in G. In fact

$$U(x,t) = \begin{pmatrix}\mathrm{sinx}\ \mathrm{cost}\\ \mathrm{sinx}\ e^{-t}\end{pmatrix},$$

is one such solution of the problem (24)-(25). Note that the above solution $U(x,t)$ is not e_2-oscillatory in G, where

$e_2 = \binom{0}{1}$.

Conclusion: In this article, a new class of oscillation criteria for impulsive vector partial conformable fractional differential equations with continuous distributed deviating arguments. We derive sufficient conditions for the H-oscillation of the solutions, using impulsive differential inequalities and the averaging technique with the Dirichlet boundary condition. We go through various examples to illustrate the improvement achieved by the results.

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