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An APOS Theory- Technoscience Framework to understand Mathematical Thinking

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Abstract
The understanding of mathematical thinking is fundamental cognitive and neuroscience phenomenon in mathematics education that can enhance both instructional and pedagogical practice for improved learning outcomes in mathematics. Several studies for the past decades have been thriving to bring out both operational and structural nature of mathematical thinking. This paper presents the novel theoretical framework as a build on to the APOS theory. The APOS theory has managed to successfully explain the mathematics thinking processes involved in the construction of a mathematics concept based on the premises of limited technology integration into mathematics education. Technology and science artefacts are now ubiquitous and constantly advancing and affecting our philosophical assumptions in mathematics education. We carried out a quasi-experiment in a mathematics laboratory involving two groups of 80 students apiece per study year taking Calculus course with differentiated teaching and learning at the University of Zimbabwe. This novel framework, APOS Theory-Technoscience, factored in technological and scientific oriented actions and processes led us to suggest an APOS-T framework as a didactic model for teaching Mathematics in Higher Education. The model could be an emerging digital pedagogy for face to face and remote mathematics instruction.

1. INTRODUCTION

Mathematical thinking is a necessary and sufficient prerequisite for positive and meaningful learning outcomes in mathematics education at any of its stages. For many decades effort to understand and unpack the operational and structural nature of mathematical thinking in mathematics education had been in place and is still progressing in social cultural, cognitive and neuroscience fronts. Piaget [1] studied the children's learning of post-secondary mathematics and developed a reflective abstraction theory which [2] and [3] further developed into the APOS theory. The fundamental tenets of the APOS theory includes the linear understanding of actions, process, objects and schemas constituted in the construction of a mathematical concept using mental mechanisms of interiorization, encapsulation, de-encapsulation, coordination, reversal, generalization and thematization.

The APOS theory has been anchored on the computer programming software to logically solve mathematical problems and to assist students encapsulate the processes behind the computer procedures and transform them into objects. However, a human based pedagogy involving activity, class discussion followed by exercises (ACE) was used to complement the computer technology. Using an APOS designed theoretical analysis, the genetic decomposition, the patterns in the students' mathematical thinking processes were able to be traced and audited for understanding and learning analytics as well as defining a mathematical learning trajectory for improved mathematics learning outcomes. However, [4] described mathematical thinking as extensive and dynamic hence its further understanding requires a review of the APOS theory holistically (linear and non-linear) and in the lens of advancing science and technology and its ubiquitous nature of availability in this current era.

Dubinsky [5] confirmed that even though the work on APOS had extensive success for many decades, there is still much room for more research in the area of pedagogical strategies which would assist students execute actions, interiorize actions into processes, condense processes into objects and put together everything into comprehensible schemas. In this study we were keen to understand mathematical thinking using APOS theory with regards to mathematics laboratory and technology integration. With this in mind, we formulated the research question: **How can an integrated pedagogical model (with APOS) be designed for use in the understanding of styudents' mathematical learning**? An optimal mix of students' interactions with technology using scientific methods were used in the mathematics pedagogics matrix to obtain the desirable mathematics learning outcomes from the resources of science and technology in the mathematics laboratory as technoscience learning space.

2. LITERATURE REVIEW

Maddox [6] and Syamsuri, Purwanto, Subanji, and Irawati [7] assert that mathematics as a discipline requires a unique way of thinking and is critical for its learning. However, [8] agrees that everyone can think mathematically but one's mathematical thinking can be developed by reflection.

Mason, Burton, and Stacey [9] explained that mathematical thinking is linked to mathematical processes such as mathematical understanding, mathematical communication and reasoning, mathematical creativity, mathematical connections, mathematical conjecturing and mathematical problem solving.

Borromeo Ferri [10] views mathematical thinking as a unique way in which the mathematics learner can present, understand and think through mathematical ideas and related connections (physical, mental and emotional) by internal thoughts or externalized depictions [11].

Mathematical thinking is fundamentally core in higher order learning required in mathematics education [12]. Pólya [13] claimed that mathematical thinking was vital in mathematics education. [14-15] also emphasized that the teaching of mathematics should target for the students' relational understanding rather instrumental understanding. Sfard [16] defined mathematical thinking as a communication tool in a mathematical language or mental cognition with both social and biological roots [17]. However, [2] referred to mathematical thinking as consciousness of thoughts or cognitive process or neural activity a learner experiences in responding to a mathematics task. It is generally a cognitive process of thinking through a mathematical problem for the purposes of executing an accurate solution. Drijvers [18] summed up mathematical thinking as an integration of problem solving, modelling and abstraction.

However, in this study, mathematical thinking is categorized as arithmetic thinking, algebraic thinking, and geometric thinking and in the realm of science and technology enhancement without loss of human intuition.

Algebraic thinking is a cognitive ability to deal with symbolic representations of abstract structures and manipulations to solve real life problems [19]. With the current availability of computer assisted systems software, algebraic thinking process can be easily enacted. Arithmetic thinking is regarded as a cognitive ability to numerically represent and manipulate objects in solving real life problems. The arithmetic thinking processes are computational processes that were efficiently done using technology such as calculators and mathematical applications rather than mentally. Geometric thinking is cognition involving properties of shapes and spatial relationships in solving real life problems. Mathematical software and graphing calculators were used as an enactment of geometric thinking process. A review of the APOS theory in a technoscientific learning space, a teaching and learning environment with ubiquitous digital resources in which mathematics, science and technology process skills are seamlessly integrated for cognition enhancement in the teaching and learning of mathematics was therefore proposed in this study. The earlier proponents of the APOS theory sought to understand mathematical thinking using APOS theory with regards to the integration of computers into the mathematics curriculum using the case of basic Calculus curriculum and other topical aspects in other disciplines other than mathematics. The natural mathematical thinking process can be made logical and robust by the use of science and technology processes without loss of intuition through enhanced visualization (diagrams & animations) and enacted virtual reality of real situations in Calculus which the human mind cannot afford.

For instance, the connections between the point, line, slope and ratio or rate of change in the conceptualization of a derivative of a function.

APOS theory in technoscientific learning space era was coined then "**APOS-T"** theory, a technoscience framework in the view of the new philosophies of cognitive science and technology and backdrop of the traditional philosophies of mathematics, mathematics education and education. Santos-Trigo, and F. Barrera-Mora [20] explained that frameworks require be reflecting on, developing and adjusting to match the technological innovations that directly influence how students learn.

The idea of viewing the world in terms of STEM or/and STEAM allow mathematics learners to envision mathematical knowledge as science and technology socially constructed from the science and technology encounter experiences while reality is mediated from the science and technology experiences without science and technology fictions. This framework is premised as both social and radical constructivist philosophy or "socio-radical constructivist philosophy".

As cited in [21], [22], [23], [24], [25], [26], [27], [28], [29-30], and [31] had extensive studies on stimulating mathematical thinking in Calculus.

[21] described mathematical thinking as an extensive and dynamic process.

Devlin [32] categorically stated that mathematics in schools is different from mathematics in colleges and universities hence mathematical thinking competence [18] is critical in mathematics education at higher order learning.

However, [33] and [34] viewed mathematical thinking as a complex connection between mathematical sense-making and knowledge and regulation of the problem-solving process. The implications are that mathematical connections are enhanced by mathematics instruction which allow the students to make sense of the nature of mathematics and how it is done [23] [35]. This study achieved this requirement by making use of Bloom's Taxonomy of mathematics instructions and the MAPLET framework. The technoscience environment of the mathematics laboratory fostered and maintained conceptual understanding of mathematics concepts and ideas through active engagement with ubiquitous digital technologies and the understanding of problem-solving process using scientific approaches.

With a wealth of evidence from earlier studies on problem solving process, absence of insights on metacognitive knowledge exist [36] and [37].

Uri [38] had characterized the three fundamental levels of mathematical thinking as arithmetic operations as innate abilities, informal mathematics in world experiences and formal mathematics learning processes. On the other hand, [39] managed to identify the five outstanding process standards for mathematical thinking as mathematical representation, mathematical reasoning and proof, mathematical communication, mathematical problem solving and mathematical connections.

Contextually, mathematical thinking was regarded as a combination of scientific thinking on using technology and the using of technology scientifically in the execution of a mathematical task. Furthermore, in this study it was acknowledged that mathematical thinking cannot be seen but can be observed in the documented cognitive actions, processes, objects and schemas depicted by students' written work or responses to questions. In this study, it was required that the mathematical thinking processes attract the students' attention to science and technology mediated problem solving processes thus embedding scientific and technological effects in the APOS theory phases. An interpretative framework such as the traditional APOS theory was invoked to seek the understanding of such mathematical thinking processes.

Dombrowski et al [40] asserts that watching out for errors in the thinking processes can immensely improve that eminence of thoughts made on our learning experiences. Conceptually, the schematic proposition of APOS-Technoscience framework as an adaption the traditional APOS theory in a technoscience learning spaces such as mathematics laboratory was as illustrated in **Figure 1.**

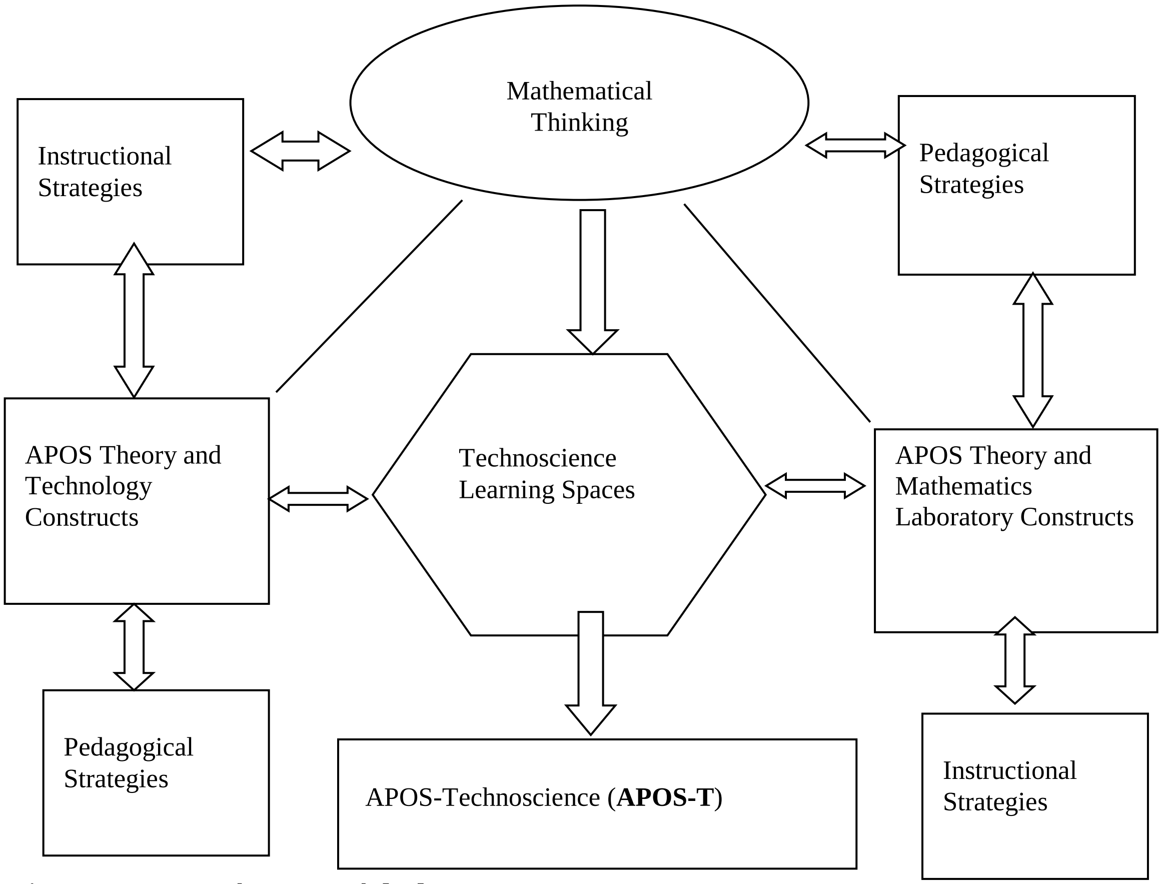

Figure 1: Conceptual Framework [41]

The interrelationships of the fundamental ideas and issues within this study were as shown in **Figure1**. Mathematical thinking is stimulated by both instructional and pedagogical strategies used in the mathematical teaching and learning process. Literature has shown that extensive research effort on how APOS theory has been successfully used to understand mathematical thinking through mental structures for conceptual constructions and use of technology but there is little evidence on how the APOS theory and Mathematics Laboratory could be similarly used for mathematical knowledge construction. Constructivist learning occurs in a social context. Allen [42] asserts that APOS theory does not preclude social interaction but argued that how or what kind of interactions that occur in a constructivist learning situation depends on the instructional design and the pedagogical strategy used.

Summit and Rickards [43] proposed that a constructivist approach to mathematics laboratory classes requires a structure for developing conceptual understanding rather than random explorations. On the other hand, Laurillard et al. [44] argued that the digital technologies are not used to optimum for effective learning and mathematics laboratories are used for remedial tutorials. This indicates that technology-rich mathematics laboratory methodology has opportunity to strike the optimum balance between use of technology and the mathematics laboratory for favourable learning outcomes.

With constantly advancing science and technology, today's mathematics learners in higher education have ubiquitous science and technology artefacts [45] for active learning and the artefacts can attribute to their mathematical thinking or assist them in their mathematical thinking.

This scenario became so imperative to the researchers to the extent of reflecting on the current status of the APOS theory in technoscience learning spaces so as to re-examine the mathematical thinking processes in the realms of science and technology attribution in a mathematics laboratory with a wider spectrum of mathematics e-resources used under the guidance of Bloom's Mathematics instruction. The study provided a window for progressing the APOS theory and unpacking the novel APOS-Technoscience Framework (**APOS-T**).

As Lerman [46] argued that while mathematics content is universal across the globe, the knowledge of mathematics education is varied, pedagogical approaches are therefore contextual to cater for socio-cultural perspectives of their acceptability [47]. Hence the APOS–Blooms Taxonomy Instruction mediated mathematics laboratory methodology used in this study is yet to be evaluated within our context and APOS-T Framework validation

3. RESEARCH METHOD

A quasi-experiment on mathematics laboratory-based APOS and Bloom taxonomy instruction, in which two groups of Calculus students at the University of Zimbabwe underwent a differentiated teaching and learning approach for a semester was used. The control group was taught using the traditional didactic pedagogy while the experimental group used the APOS-Bloom Taxonomy mediated mathematics laboratory methodology. The mathematics laboratory was a digital ecosystem with a wide spectrum of mathematics e-resources. A pre-test covering the basic concepts of Calculus I was administered to check if there are any person confounds or if any participants have certain tendencies that are likely to affect the quality of association of their performance other than the intended treatment (laboratory mathematics education). The pre-test scores were used to randomly assign students to the quasi experimental and control groups. However, using the groups' average scores as cut-off scores and previous performance it was determined whether the student was assigned to the quasi-experimental group or control group. For ethical reasons and to eliminate_bias equal numbers of the below and above group average pre-test scores for each level were allocated to the quasi experimental and control groups with the previous performance as benchmarking measure to determine poor and excellent performance while maximizing internal and external validity. Two lecturers taking Calculus courses for level one and two students (L1 and L2) participated in the study. Each lecturer had two groups characterized as experimental groups (EL1 and EL2) and control groups (CL1 and CL2) with 40 students in each group. Both groups were taught by the same lecturers and test scores for the pre-test and post- test were compared**.** We designed and developed the different laboratory-based mathematics teaching and learning instructions for the experimental groups (EL1 and EL2) matching the levels appropriately. The instructional designs were designed and developed synchronously with the research work adopting APOS (Action-Process-Object-Schema) theory in order to enhance the infusion of technology, science and mathematics thinking processes while catering for ad hoc variations that may result. The two lecturers (L1 and L2) used the curriculum instructions to teach five randomly selected topics from the Calculus modules for a period of 16 weeks or a semester. The control groups (CL1 and CL2) were taught using the traditional didactic lectures. A theoretical analysis, genetic decomposition of the students' written exercise was conducted incorporating the technoscience framework tenets into the APOS theory while the pre-test and post test scores were compared to measure the impact of the mathematics laboratory methodology, a technoscience intervention. An APOS-Technoscience (APOS-T) framework was therefore proposed.

4. RESULTS AND ANALYSIS

A qualitative reflection of the traditional APOS theory in the context of cognitive science and instructional technologies that influence human thinking occurred in the form of the proposed technoscientific framework (APOS-T) as a cognitive domain in which the epistemic notion of mathematics knowledge in this 21st century and post period was considered. The tenets of the proposed APOS-T framework unpack the ontological understanding and epistemological appropriation of the potential contribution of contemporary technoscience learning spaces such as technology-rich mathematics laboratories in further enhancing the understanding of mathematical thinking and learning outcomes in undergraduate mathematics education. Empirical evidence from the post-test scores suggested that the students who underwent the APOS theory-based mathematics laboratory teaching and learning performed better than those who were on lecture didactics (Table 1).

The basic assumption was that mathematics knowledge at undergraduate level should be regarded as concrete rather as an abstract construct [48].

The traditional APOS theory's horizon was extended to APOS theory in technoscientific learning space era and was coined "APOS-T" theory and in view of the new philosophies of cognitive science and technology, the traditional philosophies of mathematics and mathematics education. Building on the work by [28] we formulated an APOS-T Framework didactic approach illustrated in Figure 2.

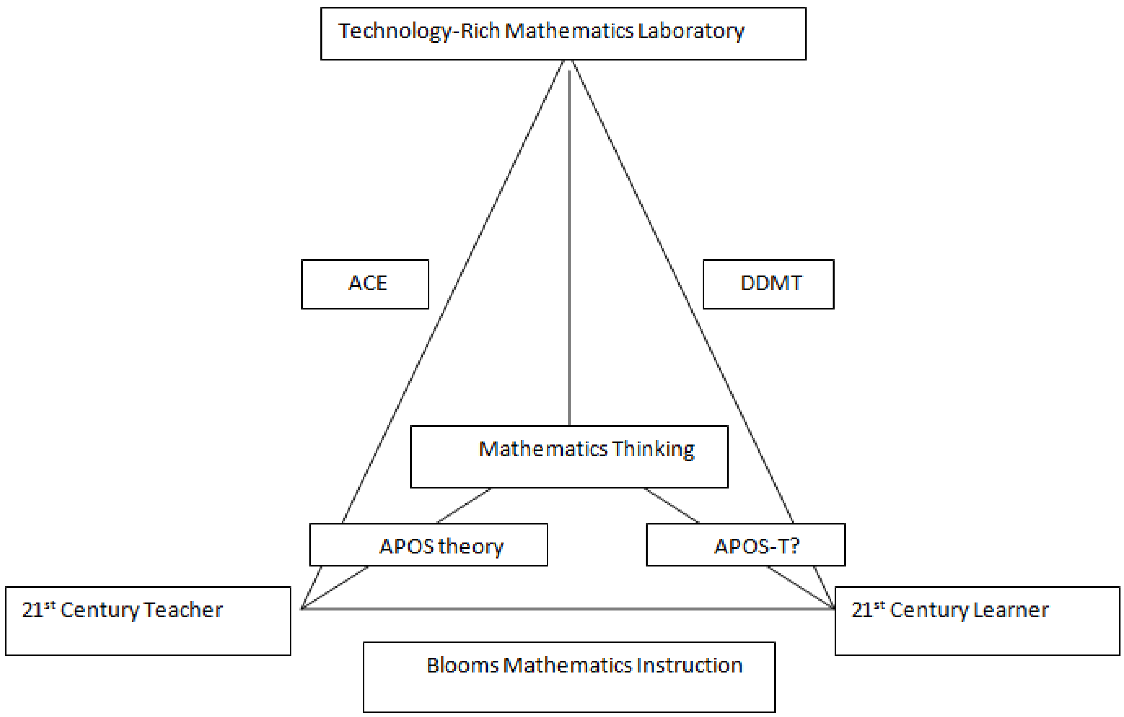

Figure 2: A Modified Didactic Tetrahedron

The Action-Process-Object-Schema-Technoscience (APOS-T) Framework, being social-radical philosophy was hereby designed for use as a build on to the traditional APOS theory as founded by Dubinsky and colleagues earlier on.

Basing on the ideas raised in Figure 2, it was acknowledged through abundant literature that the APOS theory combined with the Activity-Classroom Discussion-Exercise pedagogical strategy in a technology integrated mathematics teaching and learning, managed to explicitly explain the mathematical thinking as a precursor to mathematical understanding when cognitive neuroscience had failed to save the same purpose in mathematics education prehistorically. However, in order to fill up the pedagogical and instructional strategies gap to the APOS theory in the advent of advancing science and technology, the researchers approached the understanding of mathematical thinking in the perspective of the combination of the Blooms Taxonomy of Mathematics Instructions, Definitions of Concepts; Discovery of Concepts; Manipulations of mathematical objects and Testing of hypothesis (DDMT) from the mathematics learners' side and the traditional APOS theory and the ACE pedagogical strategy from the teachers' side in a technology-rich mathematics laboratory.

The combination of ACE, DDMT and Blooms Taxonomy provided the Mathematically Significant Pedagogical Opportunities to Build on Student Thinking (MOST) which through class discussion interactions [49] foster the conceptualisation of mathematics ideas through utilising pivotal teaching moments (PTM) as explained by [50]. Again, Leatham et al. [51] emphasized that students' mathematical thinking occusr during instruction. Ho, Huang and Looi [52] managed to apply the APOS theory and the integration of mathematical thinking and computational thinking in the design of mathematics lessons with favourable learning outcomes in a classroom environment.

Epistemologically mathematical knowledge was viewed as socially constructed from the science and technology constructs and experiences whereas mathematical reality on the other hand was observed as the applications of science and technology in mathematical concepts in the 2D or 3D visualizations and simulations or modelling of real-life phenomena or result from science or technology which could have been difficult to imagine by the human mind. Mathematical reality was accounted as what science and technology offers to mathematical objects. Reflecting on the traditional APOS theory' mental constructions re-definitions were done as follows.

In the context of this study and proposed the APOS-T framework, action was then viewed as a physical or mental transformation of mathematical objects with enhancements or affordances of science and technology in response to external situation carried out in spiral- fashion step by step with explicit focus. This means action is viewed as a cumulative and cyclic trajectory of mathematisation [53] as cited in Barnes [54] or enactment of mathematical thinking using technology or from related interpretations of technological output.

For instance, in responding to the mathematical object: Find $\sum_{n=1}^{100} n$, the students could be recalling summation from memory, or requiring an external cue, such as a step-by-step algorithm to follow on a physical scientific calculator or web-based resources with similar operations as a scientific calculator and could navigate back and forth between technological affordance and mental constructions. In this case the transformations take into account science and technology objects (computational thinking) related to the mathematical objects.

Hence the transformation was termed action-technoscience (Action-T). The basic assumption was that, to each mathematical object there is a corresponding technology or science object (computation) that will be inherent in the transformation in a technoscience mathematics learning space. This was addressing the frequent 21st mathematics learners' question: *How to do or compute a mathematical object using technology?*

Action-T was a transformation linking real objects, verbal symbols, numerical and algebraic symbols, visualisations representations (images, diagrams, graphs) in mathematical concepts using technology or scientific techniques based on the external stimulus encountered by the mathematics learners. Although it can be argued that the computer replies to a mathematical command or syntax given by the user, the requirement of communicating the mathematical idea into a computer makes it a critical and technology-embedded mathematical activity for the mathematics learners. This mathematics activity can be observed and recorded for genetic decomposition just as good as written work. Reid and Sutherland [55] assert that technologies make certain activities easy which could have been difficult such as dynamic geometry visualisation, graphical representations and complex numerical and algebraic calculations.

The action which was routinized and interiorized was therefore redefined as a technoscience process (Process-T) in which technology provided inherent logic and precision and science provided the cognition as both mathematical thinking and computational neurocognitive entities that are internally driven. The technoscience process took cognisance of the science process and technology process required in a mathematical thinking process or its enactment in the associated computational thinking.

Process-T involved the commognitive interpretations of verbal or word use, mathematics language, routinized and interiorized action-T for the processing visuals (symbols, diagrams and mathematical objects) using the affordances of technology and science. The schema remained steadfast as in the traditional APOS theory. Problem-based learning in the mathematics laboratory (service of science), computer assistance and scaffolding techniques in DDMT and ACE pedagogical strategies coupled with Blooms' instructional strategy provided further stimuli for students. The students' mental action resonated with the science and technology affordance as the ultimate expectations to their order of mathematical thinking. An object was thus regarded as an encapsulated total entity for which action and process can performed mentally or physically without and using enactment of technology or science or both. This was refinement of [56] and [57] as cited in [58] definitions of an APOS object.

It was also presumed that the APOS-T Framework assumes its innate theory characteristics from the traditional APOS theory as outlined in [59] and [60] as cited in [61] in full and extended components.

The four mental constructions encapsulation, interiorization, coordination and generalization are maintained but their occurrences are not necessarily presumed linear since the object conception with the affordances of science and technology does not necessarily precede process conception but could be systematically spiral. This would explain the transitions between the levels in the traditional APOS theory. Due to technology's build-in mathematical objects

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and processes, both object conception and process conception can concurrently occur during technological manipulations (physical structure transformation) and output step by step interpretations (mental structure transformation). The students' successful connections of such object conception and process conception can have mental encapsulation derived from technology embedded transformations if we assume technoscience learning to take common trajectory. The human-computer interactions when guided by instructions or occur by free discovery can allow the randomised mental interiorization of mathematical concepts and technology processes such that mental coordination and generalization instinctively occur in like-manner of the traditional APOS theory. The schema development was also re-explained in terms of the technoscience perspective. The intra-level was regarded as three-fold constructs involving mathematics constructs, technology constructs and science constructs where an individual can make seamless and logical relationships between them in conceptual understanding. With the inter-level, the individual is able to classify items (mathematics objects, technology objects and science objects) and relate to specific connections that can in similar examples be generalized. On the other hand, trans-level was regarded full seamless and logical mental schema of a particular mathematical concept formed from cognition enhanced by science and technology manipulations. The basic assumptions on mathematical knowledge and the hypothesis on learning [62] as cited in Maharaj [63] were maintained but with an additional condition that an individual can learn a mathematical concept indirectly with the enhancement of technology and service of science or technoscience enacted physical construction in a technoscience learning space.

This technology and science enacted physical construction tend to stimulate the real mental construction as in Piaget [1] notion. Hence science and technology artefacts do not just stimulate mathematical thinking but they can also attribute to salient aspects of thinking such logical computations, precision and accuracy in actions and processes involved in the human computer interactions or mathematisation.

The framework has a potential of providing understanding of mathematical, pedagogical and instructional situations that add to further understanding of mathematical thinking in contemporary undergraduate lectures offered in technoscience learning spaces at higher education.

Johnson, Keller and Fukawa-Connelly [64] as cited in [65] confirmed that mathematics instructions that favoured active learning had remained static at the undergraduate level lecture. On the other hand, in the March 2016 issue of The Teaching Professor newsletter, Scott et al. [66] acclaimed that undergraduate students who undergo the traditional lectures as pedagogical and instructional strategy were 1.5 times more likely to fail than their counterparts engaged in motivating and interactive active learning techniques. Therefore, this theoretical basis for ways of understanding mathematical thinking at tertiary level and in this particular instance in Calculus teaching and learning in the new era of technoscience learning space need constant improvements and trending. It is however fundamental to mention that the proposed APOS-T Framework seeks to motivate some reflection of the traditional APOS theory in view of the technoscience learning spaces without downgrading it but to improve it further. A cognitive development model was created for the APOS-T framework as in Figure 3.

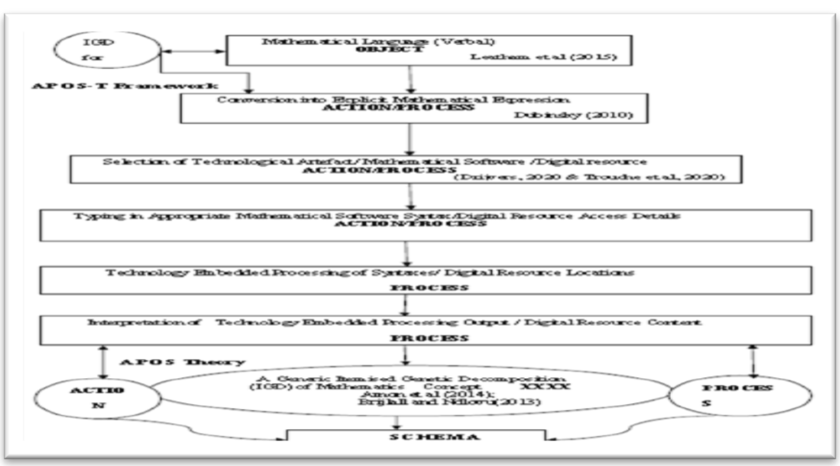

Figure 3: A cognitive development model for technoscience learning space

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4.1. A cognitive development model for technoscience learning space

From Figure 3, a technoscience learning space such as a digitally well-resourced mathematics laboratory, students' mental constructions begin in commognitive trajectory [51]. Firstly, students faced with a mathematical problem or situation need to extract and convert the mathematical language (verbal expressions) into some explicit mathematical expressions [62]. Secondly, the students then use their technoscience acumen to select the most appropriate technological artefact or mathematical software or online digital resource for use in the accomplishment of a particular mathematical task. Thirdly, the individuals should issue some appropriate mathematical software syntaxes or online digital resource access details to allow technologically embedded processing which otherwise would a difficult to do manually. These actions and processes are interiorised by the students and later on reflected upon for use beyond the given mathematical situation or problem with or without necessarily engaging technology. Fourthly, with this technology integration in the computational and mathematical thinking processes, the affordance and enhancement of technology [30] ; [67] the interpretations of the output set the initial trajectory into the meta-cognitive path where the mental constructions(actions , processes, objects) and related triad mechanisms [59] come into usual play for schema development. Lastly, based on the step-by-step interpretations of the output, the itemized genetic decomposition (IGD) as conducted by [68] could be conducted. This IGD coupled with thematisation and coordination processes by the students through exposure and consolidation could assist the students' mathematical thinking as in the proposed APOS-T Framework. The empirical evidence for applicability of the framework in the classroom practice was provided from the comparison of the test scores of the control and experimental groups in the form the analysis of variance.

4.2. Analysis of Variance of Post Scores of Control and Experimental Groups

The effectiveness of the intervention between and within the groups was compared using the analysis of variance (ANOVA) in Table 1.

Table 1: Analysis of Variance.

ANOVA							
Source of Variations		Sum of Squares	Df	Mean Square	F	P-Value.	F Crit
Test Score: **(Combined)** Control Group Experimental Group	Between Groups	2600.16	1	2600.16	32.87	4.87×10^{-8}	3.90
	Within Groups	12498.69	158	79.11			
	Total	15098.84	159				
Level I: Control Group Experimental Group	Between Groups	1051.25	1	1051.250	11.28	1.22×10^{-3}	3.96
	Within Groups	7271.95	78	93.23			
	Total	8323.20	79				
Level II: Control Group Experimental Group	Between Groups	1575.31	1	1575.31	23.63	5.92×10^{-6}	3.96
	Within Groups	5199.58	78	66.66			
	Total	6774.89	79				

There were some statistically significant mean score differences between the control and experimental groups for each level, $F(1, 78) = 11.28$ with p-value $=1.22 \times 10^{-3}$ for Level I and $F(1, 78) = 23.63$ with p-value $=5.92 \times 10^{-6}$ for

Level II respectively. This implied that there were some significant differences in the performance of control and experimental groups under this study.

4. CONCLUSION

A motion is set up as a provocation for further discussions on the technoscience oriented reflection and analysis of the APOS theory both in theoretical and practical realms to transform it into APOS-T, a technoscience framework. The integrated model was used in this study to understand students' learning in mathematics. There is a wider acceptance of the mathematics laboratory methodology as it offers inquiry based educational strategies associated with clear mathematical knowledge gains and improved course performance at undergraduate level. The potential for instructional practice shift and improved learning outcomes in undergraduate Calculus exists with optimal mixes of pedagogical strategies to the mathematics laboratory-based APOS and Bloom instruction but it remains as both a professional and institutional obligation.

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