Homotopy Continuation Method to Solve System of Nonlinear equations

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Abstract:
Numerical method is a mathematical tool designed to solve the problems arises in the field of mathematics, engineering, computer science and physical sciences[12, 11, 9]. There are several methods to solve the system of linear and nonlinear equations, such as iterative methods, approximation methods and interpolation methods. In this paper we use Homotopy Continuation method to solve system of nonlinear equations[11]. This method reduces a difficult problem into a simpler problem and helps in finding the solution of the original problem[3]. It is both locally and globally convergent and numerically stable[3, 8]. It is used to calculate critical loading condition with nose curve, stability analysis in the global minimization of the Gibb's free energy and convergence analysis in multi-phase, multi-reaction equilibrium systems. Homotopy Continuation method is different from iterative methods. The advantage of this method is that one can find the solution of the problem by choosing an arbitrary initial value. Unlike the iterative methods, where one has to have enough knowledge about the location of the root to find the initial value[3]. If the initial value is closer to the solution then the method will be more efficient, otherwise the solution will diverge from the actual solution. In this method the system of nonlinear equations are reduced to system of linear differential equations, which are solved by Runge-Kutta methods. We have compared the Continuation method with Broyden's method and found that the results are more accurate in Continuation method.
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1 Introduction

Nonlinear system is a set of two or more nonlinear equations having the same set of variables. There are many iterative process for solving system of nonlinear equations. Homotopy continuation methods (HCM) in con-junction with a classical numerical method used to determine roots of nonlinear algebraic equations for better performance of the classical numerical method[7]. Kalaba et. al. have solved system of nonlinear equations by adaptive Homotopy Continuation method and allowed the continuation parameters to take complex values and to grow adaptively in the complex plane to avoid singular points[13]. Verschelde have solved polynomial system for numerical approximations using Homotopy Continuation method to all isolated complex solution[17]. Hafizudin et. al. have solved several polynomial equations using Osrowski Homotopy Continuation method and Ostrowski method and hence shown that Ostrowski Homotopy Continuation method converge faster and able to solve divergence problems whereas, classical Ostrowski method becomes inconsistent and converge slowly[7]. Burden et. al.[3] stated that Newton's method requires an accurate initial approximation to

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ensure convergence. Further, Chapra et. al.[4] stated that convergence of Newton's method depends on the nature of the function and accuracy of initial guess. Hafizudin et. al. have applied newton Homotopy Continuation method, secant Homotopy Continuation method and Adomian Homotopy Continuation method for nonlinear algebraic equations to check for greater accuracy and applicability and it is seen that newton Homotopy Continuation method is more accurate than the other methods[8]. Zuria et. al. have done the comparitive study of classical newton's method and newton Homotopy Continuation method and they have shown that newton Homotopy Continuation method can solve the can solve the divergence problem and has lesser iterations rather than classical newton's method[18] when the selection of initial value is inappropriate. But, in Newton method and many more iterative techniques, the convergence depends only on the good choice of approximation which is a drawback of these methods. To overcome the local convergence of the above mentioned methods, we use Homotopy Continuation Method which is classified as imbedding or increment loading methods[8]. This method is advantageous because this produces a solution in a large range of the independent variable thus we can analyse complete behaviour of the solution[5]. This method was known since $1930s$. But in 1972, B.C. Eaves developed the concept of Homotopy. Recently, Morgan worked on the development of Homotopy concepts particularly for the algebraic equations. According to Palancz et. al. Homotopy Continuation Method deforms the known roots of the initial system into the roots of the final system. Here Homotopy method helps us in converting the nonlinear equations into a simpler algebraic equation which can be further solved by usual known methods[15, 3]. We use Runge-Kutta fourth order method in comparison with the Euler method to solve the equations because Runge-Kutta fourth order method is superior than other methods[16, 2].

2 Methodology

Consider the equation of the form

$$F(x) = 0. \tag{1}$$

Let x_0 be the initial approximation of equation (1). Let x^* be the unknown solution of

$$F(x^*) = 0. \tag{2}$$

Let $\lambda \in [0,1]$ be the parameter. Now, Suppose x_0 be the initial approximation to $F(x^*) = 0$. Define the Homotopy G as

$$G: [0,1] \times \mathbb{R}^n \to \mathbb{R}^n \text{ given by,}$$

$$G(\lambda, x) = \lambda F(x) + (1 - \lambda)[F(x) - F(x_0)], \tag{3}$$

$$G(\lambda, x) = \lambda F(x) + F(x) - F(x_0) - \lambda F(x) + \lambda F(x_0), \tag{4}$$

$$G(\lambda, x) = \lambda F(x) + (\lambda - 1)F(x_0). \tag{5}$$

Now put $\lambda = 0$ in (5)

$$G(0, x) = F(x) - F(x_0) = 0. \tag{6}$$

Again put $\lambda = 1$ in (5)

$$G(1, x) = F(x) = 0. \tag{7}$$

G is the Homotopy function between equation (6) and (7).

Further, To find x^* which is a solution of $F(x) = 0$. Suppose $x(\lambda)$ be the unique solution of

$$G(\lambda, x) = 0. \tag{8}$$

The set $\{x(\lambda)\}$, $0 \leq \lambda \leq 1$ is a curve in $\mathbb{R}^n$ parametrized by λ. Along the curve we get a sequence corresponding to $\{x(\lambda_k)\}_{k=0}^{m}$ where $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m = 1$ are subintervals.

If $\lambda \to x(\lambda)$ and G are differentiable, then differentiate equation (8) with respect to $'\lambda'$, we get

$$\frac{\partial G}{\partial \lambda}(\lambda, x(\lambda)) + \frac{\partial G}{\partial x}(\lambda, x(\lambda))x^{'}(\lambda) = 0, \tag{9}$$

$$\frac{\partial G}{\partial x}(\lambda, x(\lambda))x^{'}(\lambda) = -\frac{\partial G}{\partial \lambda}(\lambda, x(\lambda)), \tag{10}$$

$$x^{'}(\lambda) = -\left[\frac{\partial G}{\partial x}(\lambda, x(\lambda))\right]^{-1}\frac{\partial G}{\partial \lambda}(\lambda, x(\lambda)). \tag{11}$$

Equation (11) is the system of differential equations.

To solve equation (11), first we need to find $\frac{\partial G}{\partial x}$ and $\frac{\partial G}{\partial \lambda}$.

Consider

$$\frac{\partial G}{\partial x}(\lambda, x(\lambda)) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x(\lambda)) & \frac{\partial f_1}{\partial x_2}(x(\lambda)) \ldots \frac{\partial f_1}{\partial x_n}(x(\lambda)) \\ \frac{\partial f_2}{\partial x_1}(x(\lambda)) & \frac{\partial f_2}{\partial x_2}(x(\lambda)) \ldots \frac{\partial f_2}{\partial x_n}(x(\lambda)) \\ \vdots & \\ \frac{\partial f_n}{\partial x_1}(x(\lambda)) & \frac{\partial f_n}{\partial x_2}(x(\lambda)) \ldots \frac{\partial f_n}{\partial x_n}(x(\lambda)) \end{bmatrix}, \tag{12}$$

$$\frac{\partial G}{\partial x}(\lambda, x(\lambda)) = J(x(\lambda)), \tag{13}$$

$$\frac{\partial G}{\partial x} = J. \tag{14}$$

Again consider

$$\frac{\partial G}{\partial \lambda}(\lambda, x(\lambda)) = \frac{\partial}{\partial \lambda}[F(x(\lambda)) + (\lambda - 1)F(x_0)], \tag{15}$$

$$\frac{\partial G}{\partial \lambda}(\lambda, x(\lambda)) = F(x_0). \tag{16}$$

Substitute equations (14) and (16) in (11), we get

$$x^{'}(\lambda) = -[J(x(\lambda))]^{-1}F(x_0), \tag{17}$$

where $0 \le \lambda \le 1$.

In general we can write it as

$$\begin{aligned} \frac{dx_1}{d\lambda} &= \phi_1(\lambda, x_1, x_2, \ldots, x_n), \\ \frac{dx_2}{d\lambda} &= \phi_2(\lambda, x_1, x_2, \ldots, x_n), \\ &\vdots \\ \frac{dx_n}{d\lambda} &= \phi_n(\lambda, x_1, x_2, \ldots, x_n). \end{aligned} \tag{18}$$

Equation (18) is written in the matrix form as

$$\begin{bmatrix} \phi_1(\lambda, x_1, x_2, \ldots, x_n) \\ \phi_2(\lambda, x_1, x_2, \ldots, x_n) \\ \vdots \\ \phi_n(\lambda, x_1, x_2, \ldots, x_n) \end{bmatrix} = -[J(x_1, x_2, \ldots, x_n)]^{-1}\begin{bmatrix} f_1(x_0) \\ f_2(x_0) \\ \vdots \\ f_n(x_0) \end{bmatrix}. \tag{19}$$

Equation (19) can be solved using Euler's method, Runge Kutta second order and Runge Kutta fourth order method. To solve equation (19), let $N > 0$ and $h = \frac{b-a}{N} = \frac{1-0}{N}$.
Let the [0,1] is partitioned into N subintervals with gridpoints. Let $\lambda_j = jh$ for each $j = 0,1,\dots,N$. Let w_{ij} be the approximation to $x_i(\lambda_j)$.
Set the initial conditions

$$\begin{aligned} w_{1,0} &= x_1(0), \\ w_{2,0} &= x_2(0), \\ &\vdots \\ w_{n,0} &= x_n(0). \end{aligned} \tag{20}$$

And calculate $\{w_{1,j}, w_{2,j}, \dots, w_{n,j}\} = w_j$. Now, to calculate $w_{1,j+1}, w_{2,j+1}, \dots, w_{n,j+1}$, we use the following two methods.

1. **Euler method**

$$\begin{aligned} w_{i,j+1} &= w_{i,j} + h[-J(w_j)]^{-1} F(x_0), \\ or\ x(\lambda_{j+1}) &= x(\lambda_j) + h[-J(w_j)]^{-1} F(x_0). \end{aligned} \tag{21}$$

for $j = 0,1,\dots,N$

2. **Runge Kutta Fourth Order Method**

$$k_1 = h[-J(w_j)]^{-1} F(x_0), \tag{22}$$

$$k_2 = h\left[-J\left(w_j + \frac{k_1}{2}\right)\right]^{-1} F(x_0), \tag{23}$$

$$k_3 = h\left[-J\left(w_j + \frac{k_2}{2}\right)\right]^{-1} F(x_0), \tag{24}$$

$$k_4 = h[-J(w_j + k_3)]^{-1} F(x_0), \tag{25}$$

and

$$\begin{aligned} w_{i,j+1} &= w_{i,j} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \\ or\ x(\lambda_{j+1}) &= x(\lambda_j) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4). \end{aligned} \tag{26}$$

for $j = 0,1,\dots,N$

The follwing theorem says under what circumstances the Homotopy Continuation Method is feasible[3, 15, 6, 14, 9, 2].

3 THEOREM

If $F(x)$ is continuously differentiable for $x \in \mathbb{R}^n$ and $J(x)$ be non-singular jacobian matrix $\forall x \in \mathbb{R}^n$. Then there exist a constant M, Such that $\| J(x)^{-1} \| \leq M, \forall x \in \mathbb{R}^n$. Then for any x_0 in $\mathbb{R}^n$, There exist a unique function $x(\lambda)$, Such that

$$G(\lambda, x(\lambda)) = 0. \forall \lambda \in [0,1]. \tag{27}$$

Moreover, $x(\lambda)$ is continuously differentiable and $x'(\lambda) = -[J(x(\lambda))]^{-1} F(x_0)$, Foreach $\lambda \in [0,1]$.

Proof: Let the function $G: [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ defined by $G(\lambda, x) = \lambda F(x) + (1-\lambda)[F(x) - F(x_0)]$ is a Homotopy function. Let x_0 be a initial approximation. If all the properties of F are

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replaced by the properties of G then there exists a unique continuously differentiable function such that $G(\lambda, x) = 0$. Since G^1 is non singular it is also refers to regular imbedding otherwise, if it is singular it refers to singular imbedding for F to $x_0 \in \mathbb{R}^n$[5]. In that case, we consider another function $G: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ defined by $G(x,t) = H(x,t)$ and thus the requirement to regularity of H' is replaced by the properties of linearly independent decomposition from $G^1(u) = (H', H'')$.

3.1 Example 1:

Consider the system of non linear transcendental equations

$$\begin{aligned} f_1(x_1,x_2,x_3) &= 3x_1 - \cos(x_2x_3) - 0.5, \\ f_2(x_1,x_2,x_3) &= x_1^2 - 81(x_2+0.1)^2 + \sin(x_3) + 1.06, \\ f_3(x_1,x_2,x_3) &= e^{(-x_1x_2)} + 20x_3 + \frac{10\pi-3}{3}. \end{aligned} \tag{28}$$

Initial approximation $x_0 = (0,0,0,)^t, N = 4$. The actual solution of the above system is $(0.5,0,-0.52359877)^t$.

Solution: We have,

$$J(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix}. \tag{29}$$

Differentiate equation (28) with respect to the variables x_1, x_2, x_3, we get

$$J(x) = \begin{bmatrix} 3 & x_3\sin(x_2x_3) & x_2\sin(x_2x_3) \\ 2x_1 & -162(x_2+0.1) & \cos(x_3) \\ -x_2e^{(-x_1x_2)} & -x_1e^{(-x_1x_2)} & 20 \end{bmatrix}. \tag{30}$$

Given the initial approximation

$$x_0 = (0,0,0)^t. \tag{31}$$

Using equation (31) in equation (28), we get

$$F(x_0) = \begin{bmatrix} -1.5 \\ 0.25 \\ \frac{10\pi}{3} \end{bmatrix}. \tag{32}$$

$$h = 0.25. \tag{33}$$

Since from equation (20), we have

$$w(0) = (0,0,0)^t. \tag{34}$$

By Homotopy Continuation Method we have

$$x'(\lambda) = -\left[\frac{\partial G}{\partial x}(\lambda, x(\lambda))\right]^{-1} \frac{\partial G}{\partial \lambda}(\lambda, x(\lambda)). \tag{35}$$

We get the system of differential equations from equation(35), which are solved using Runge Kutta Fourth Order Method and Euler Method. The numerical solutions obtained are compared.

<u>Runge Kutta Fourth Order Method</u>

For $N = 1, j = 0$.

Equation (22) becomes,

$$k_1 = h\left[-J(w^{(0)})\right]^{-1} F(x_0). \tag{36}$$

Now substitute equation (32), equation (33), and equation (34) in equation (36), we get

$$k_1 = 0.25\begin{bmatrix} -3 & 0 & 0 \\ 0 & 16.2 & -1 \\ 0 & 0 & -20 \end{bmatrix}^{-1} \begin{bmatrix} -1.5 \\ 0.25 \\ \frac{10\pi}{3} \end{bmatrix}, \tag{37}$$

$$k_1 = (0.125, -0.004222203327, -0.130899693)^t, \tag{38}$$

$$\frac{k_1}{2} = (0.0625, -0.002111101664, -0.065449846)^t. \tag{39}$$

Equation (23) becomes,

$$k_2 = h\left[-J\left(w_0 + \frac{k_1}{2}\right)\right]^{-1} F(x_0). \tag{40}$$

Now substitute equation (32), equation (33), and equation (39) in equation (40), we get

$$k_2 = 0.25\begin{bmatrix} -3 & 9.04328889 \times 10^{-6} & -2.916936157 \times 10^{-7} \\ -0.125 & 15.85800153 & -0.997858923 \\ -0.002111380229 & 0.062508247 & -20 \end{bmatrix}^{-1} \begin{bmatrix} -1.5 \\ 0.25 \\ \frac{10\pi}{3} \end{bmatrix}, \tag{41}$$

$$k_2 = (0.125000002, -0.003311761792, -0.13092324)^t, \tag{42}$$

$$\frac{k_2}{2} = (0.062500001, -0.001655880896, -0.06546162)^t. \tag{43}$$

Equation (24) becomes,

$$k_3 = h\left[-J\left(w_0 + \frac{k_2}{2}\right)\right]^{-1} F(x_0). \tag{44}$$

Now substitute equation (32), equation (33), and equation (43) in equation (44), we get

$$k_3 = 0.25\begin{bmatrix} -3 & 7.095820034 \times 10^{-6} & 1.794991935 \times 10^{-7} \\ -0.125000002 & 15.93174729 & -0.997858153 \\ -0.001656052276 & 0.062506469 & -20 \end{bmatrix}^{-1} \begin{bmatrix} -1.5 \\ 0.25 \\ \frac{10\pi}{3} \end{bmatrix}, \tag{45}$$

$$k_3 = (0.124999984, -0.003296244627, -0.130920346)^t. \tag{46}$$

Equation (25) becomes,

$$k_4 = h[-J(w_0 + k_3)]^{-1} F(x_0). \tag{47}$$

Now substitute equation (32), equation (33), and equation (46) in equation (47), we get

$$k_4 = 0.25\begin{bmatrix} -3 & 5.649808273 \times 10^{-5} & 1.422479449 \times 10^{-6} \\ -0.249999968 & 15.66600837 & -0.991442165 \\ -0.00329760306 & 0.125051498 & -20 \end{bmatrix}^{-1} \begin{bmatrix} -1.5 \\ 0.25 \\ \frac{10\pi}{3} \end{bmatrix}, \tag{48}$$

$$k_4 = (0.124999894, -0.002302067618, -0.130934697)^t. \tag{49}$$

Then

$$w_1 = w_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4). \tag{50}$$

Now substitute equations (38), (42), (46), (49) in equation (50), we get

$$w_1 = \begin{bmatrix} 0.124999977 \\ -0.003290047297 \\ -0.13092026 \end{bmatrix}. \quad (51)$$

continuing like this, we get

$$w_2 = \begin{bmatrix} 0.249999774 \\ -0.004507399882 \\ -0.26185576 \end{bmatrix}, \quad (52)$$

$$w_3 = \begin{bmatrix} 0.374999701 \\ -0.003430351774 \\ -0.392763439 \end{bmatrix}, \quad (53)$$

$$w_4 = \begin{bmatrix} 0.0.5 \\ -0.1.3065 \times 10^{-8} \\ -0.523598772 \end{bmatrix}. \quad (54)$$

The values obtained in equation (54) are the approximate solutions of the given nonlinear system.

Euler Method

For $\boldsymbol{N} = \mathbf{1}, \boldsymbol{j} = \mathbf{0}$.

$$h = 1. \quad (55)$$

Equation (21) becomes,

$$w_1 = h\left[-J\,(w^{(0)})\right]^{-1} F(x_0). \quad (56)$$

Now substitute equation (32), equation (34), and equation (55) in equation (56), we get

$$w_1 = 0.25 \begin{bmatrix} -3 & 0 & 0 \\ 0 & 16.2 & -1 \\ 0 & 0 & -20 \end{bmatrix}^{-1} \begin{bmatrix} -1.5 \\ 0.25 \\ \frac{10\pi}{3} \end{bmatrix}, \quad (57)$$

$$w_1 = \begin{bmatrix} 0.5 \\ -0.016888813 \\ -0.523598775 \end{bmatrix}. \quad (58)$$

The values obtained in equation (58) are the approximate solutions of the given nonlinear system. The results are shown in the Table: 1.1

Comparison between Homotopy Continuation Method and Broyden's Method

The system of equations in (28) are solved by Homotopy Continuation Method, using Runge Kutta Fourth order for system of differential equations and also the system of equations in (28) are solved by Broyden's method. The results are shown in the Table: 1.2

3.2 Example 2:

Consider the system of non linear algebraic equations

$$f_1(x_1, x_2, x_3) = 15x_1 + x_2^2 - 4x_3 - 13,$$
$$f_2(x_1, x_2, x_3) = x_1^2 + 10x_2 - x_3 - 11, \quad (59)$$

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$$f_3(x_1, x_2, x_3) = x_2^3 - 25x_3 + 22.$$

Initial approximation $x_0 = (1,1,1)^t$. The system of differential equations obtained by applying Homotopy Continuation method to equation (??) are solved using Runge Kutta Fourth Order Method and Euler Method. The numerical solutions obtained are compared. The results are shown in the Table: 1.3

4 Results and Discussions

We have solved a system of nonlinear equations by Homotopy Continuation method using Runge Kutta method of fourth order and Euler method. From the Table: 1.1 and Table: 1.3, we observe that the values obtained from the Runge Kutta Fourth order method are more accurate than the values obtained from the Euler method. Also from the Table: 1.2, we analyse that the Homotopy Continuation method converges faster than the Broyden's method.

5 Tables

Table: 1.1 <u>Comparison between Runge Kutta Fourth Order Method and Euler Method</u>

Iterations	Runge Kutta Method	Iterations	Euler Method
04	$(0.5, -0.13065 \times 10^{-8}, -0.523598772)^t$	01	$(0.5, -0.016888813, -0.523598775)^t$

Table: 1.2 <u>Comparison between Homotopy Continuation Method and Broyden's Method</u>

Iterations	Homotopy Continuation Method	Iterations	Broyden's Method
01	$(0.125000, -0.003290, -0.130920)^t$	01	$(0.499987, -0.001454, -0.524023)^t$
02	$(0.250000, -0.004507, -0.261856)^t$	02	$(0.500001, 0.000130, -0.523595)^t$
03	$(0.375000, -0.003430, -0.392763)^t$	03	$(0.500000, -0.000002, -0.523599)^t$
04	$(0.500000, 0.000000, -0.523599)^t$	04	$(0.500000, -0.000000, -0.523599)^t$
05		05	$(0.500000, 0.000000, -0.523599)^t$

Table: 1.3 <u>Comparison between Runge Kutta Fourth Order Method and Euler Method</u>

Iterations	Runge Kutta Method	Iterations	Euler Method
01	$(1.036400, 1.085707, 0.931191)^t$	01	$(1.036649, 1.085699, 0.930284)^t$

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