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NUMERICAL INTEGRATION OF A SINGLE CLASS OF DIFFERENTIAL EQUATIONS WITH MINOR MODIFICATIONS

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ABSTRACT:

A terminal boundary-value technique is presented for solving singularly perturbed delay differential equations, the solutions of which exhibit layer behaviour. By introducing a terminal point, the original problem is divided into inner and outer region problems. An implicit terminal boundary condition at the terminal point was determined. The outer region problem with the implicit boundary condition was solved and produces an explicit boundary condition for the inner region problem. Then, the modified inner region problem (using the stretching transformation) is solved as a two-point boundary value problem. The second-order finite difference scheme was used to solve both the inner and outer region problems. The proposed method is iterative on the terminal point. To validate the efficiency of the method, some model examples were solved. The stability and convergence of the scheme was also investigated.

1. INTRODUCTION

`The singularly perturbed delay differential equations with small shift arise very frequently in the modeling of various physical and biological phenomena, for example, micro scale heat transfer, hydrodynamics of liquid helium, second-sound theory, thermoelasticity, diffusion in polymers, reaction-diffusion equations, stability, control of chaotic systems 8, a variety of models for physiological processes or diseases and so forth. Hence in the recent times, many researchers have been trying to develop numerical methods for solving these problems. Amiraliyev and Cimen 10presented numerical method comprising a fitted difference scheme on

a uniform mesh to solve second-order delay differential equations. Lange and Miura gave an asymptotic approach for a class of boundary-value problems for linear second-order differential-difference equations. Kadal bajoo and Sharma presented numerical approaches to solve singularly perturbed differential-difference equations, which contains negative shift in the convention term i.e., in the derivative term. Lange and Miura 16 considered the boundary value International Journal of Differential Equations problem for a singularly perturbed nonlinear differential difference equation with shift and discussed the existence and uniqueness of their solutions. Furthermore, Kadalbajoo and Sharma have discussed the numerical solution of the singularly perturbed nonlinear differential equations with small negative shifts. In this paper, we have presented a numerical integration method for solving a class of singularly perturbed delay differential equations with small shift. First, the secondorder singularly perturbed delay differential equation is replaced by an asymptotically equivalent first-order delay differential

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equation. Then we employed Simpson's rule and linear interpolation to get three-term recurrence relation which is solved easily by discrete invariant imbedding algorithm. The method is demonstrated by implementing it on several linear and nonlinear model examples by taking various values for the delay and perturbation parameters.

2. DESCRIPTION OF METHOD:

Consider a class of singularly perturbed boundary value problems of the following form:

$$Ly \equiv \varepsilon y''(x) + a(x)\, y'(x-\delta) + b(x)\, y(x) = f(x), \quad 0 \le x \le 1, \tag{2.1}$$

with the interval and boundary conditions

$$y(0) = \alpha, \quad -\delta \le x \le 0, \tag{2.2a}$$

$$y(1) = \beta, \tag{2.2b}$$

The transition from is admitted, because of the condition that is small, . This replacement is significant from the computational point of view. Further details on the validity of this transition can be found.

By using Simpson's rule to evaluate the integral, we get

$$\begin{aligned} y_{i+1} - y_i = {} & p_{i+1} y(x_{i+1}-\delta) - p_i y(x_i - \delta) \\ & - \frac{h}{6}\left(p_i' y(x_i-\delta) + 4p_{i+1/2}' y\left(x_{i+1/2}-\delta\right) + p_{i+1}' y(x_{i+1}-\delta)\right) \\ & + \frac{h}{6}\left(q_i y(x_i-\sqrt{\varepsilon}) + 4q_{i+1/2} y\left(x_{i+1/2}-\sqrt{\varepsilon}\right) + q_{i+1} y(x_{i+1}-\sqrt{\varepsilon})\right) \\ & + \frac{h}{6}\left(r_i y_i + 4r_{i+1/2} y_{i+1/2} + r_{i+1} y_{i+1}\right) + \frac{h}{6}\left(s_i + 4s_{i+1/2} + s_{i+1}\right). \end{aligned}$$

By the means of Taylor series expansion and then by approximating by linear interpolation, we get

$$y(x_i-\delta) = y(x_i) - \delta y'(x_i) = y_i - \delta\left(\frac{y_i - y_{i-1}}{h}\right) = \left(1-\frac{\delta}{h}\right) y_i + \frac{\delta}{h} y_{i-1},$$

$$y(x_{i+1}-\delta) = y(x_{i+1}) - \delta y'(x_{i+1}) = y_{i+1} - \delta\left(\frac{y_{i+1} - y_i}{h}\right) = \left(1-\frac{\delta}{h}\right) y_{i+1} + \frac{\delta}{h} y_i,$$

$$y(x_i-\sqrt{\varepsilon}) = y(x_i) - \sqrt{\varepsilon} y'(x_i) = y_i - \sqrt{\varepsilon}\left(\frac{y_i - y_{i-1}}{h}\right) = \left(1-\frac{\sqrt{\varepsilon}}{h}\right) y_i + \frac{\sqrt{\varepsilon}}{h} y_{i-1},$$

$$y(x_{i+1}-\sqrt{\varepsilon}) = y(x_{i+1}) - \sqrt{\varepsilon} y'(x_{i+1}) = y_{i+1} - \sqrt{\varepsilon}\left(\frac{y_{i+1} - y_i}{h}\right) = \left(1-\frac{\sqrt{\varepsilon}}{h}\right) y_{i+1} + \frac{\sqrt{\varepsilon}}{h} y_i.$$

In similar way,

$$y\left(x_{i+1/2}-\delta\right)=y\left(x_{i+1/2}\right)-\delta y'\left(x_{i+1/2}\right)=y_{i+1/2}-\delta\left(\frac{y_{i+1}-y_i}{h}\right)=y_{i+1/2}-\frac{\delta}{h}y_{i+1}+\frac{\delta}{h}y_i.$$

Finally, making use of rearranging as three-term recurrence relation, we get

$$E_i y_{i-1}-F_i y_i+G_i y_{i+1}=H_i,$$

for $i=1,2,\ldots,N-1$, where

$$E_i=\frac{\delta}{h}\left(p_i+\frac{h}{6}p_i'\right)-\frac{\sqrt{\varepsilon}}{6}q_i-\frac{4h}{6}\left(-p_{i+1/2}'+q_{i+1/2}+r_{i+1/2}\right)\left(-\frac{\delta}{8h}\left(p_i-p_{i+1}\right)+\frac{\sqrt{\varepsilon}}{8}q_i\right),$$

$$\begin{aligned}F_i={}&1+\frac{\delta}{h}\left(p_{i+1}-\frac{h}{6}p_{i+1}'\right)-\left(1-\frac{\delta}{h}\right)\left(p_i+\frac{h}{6}p_i'\right)-\frac{4\delta}{6}p_{i+1/2}'\\&+\frac{h}{6}\left(1-\frac{\sqrt{\varepsilon}}{h}\right)q_i+\frac{4\sqrt{\varepsilon}}{6}q_{i+1/2}+\frac{\sqrt{\varepsilon}}{6}q_{i+1}+\frac{h}{6}r_i+\frac{4h}{6}\left(-p_{i+1/2}'+q_{i+1/2}+r_{i+1/2}\right)\\&\times\left(\frac{1}{2}+\frac{\delta}{8h}\left(p_i-p_{i+1}\right)-\frac{1}{8}\left(1-\frac{\delta}{h}\right)\left(p_i-p_{i+1}\right)+\frac{h}{8}\left(1-\frac{\sqrt{\varepsilon}}{h}\right)q_i+\frac{h}{8}r_i-\frac{\sqrt{\varepsilon}}{8}q_{i+1}\right),\end{aligned}$$

$$\begin{aligned}G_i={}&1-\left(1-\frac{\delta}{h}\right)\left(p_{i+1}-\frac{h}{6}p_{i+1}'\right)-\frac{4\delta}{6}p_{i+1/2}'-\frac{h}{6}\left(1-\frac{\sqrt{\varepsilon}}{h}\right)q_{i+1}\\&+\frac{4\sqrt{\varepsilon}}{6}q_{i+1/2}-\frac{h}{6}r_{i+1}-\frac{4h}{6}\left(-p_{i+1/2}'+q_{i+1/2}+r_{i+1/2}\right)\\&\times\left(\frac{1}{2}+\frac{1}{8}\left(1-\frac{\delta}{h}\right)\left(p_i-p_{i+1}\right)-\frac{h}{8}\left(1-\frac{\sqrt{\varepsilon}}{h}\right)q_{i+1}-\frac{h}{8}r_{i+1}\right),\end{aligned}$$

$$H_i=\frac{h}{6}\left(s_i+4s_{i+1/2}+s_{i+1}\right)+\frac{4h}{6}\left(-p_{i+1/2}'+q_{i+1/2}+r_{i+1/2}\right)\left(\frac{h}{8}\left(s_i-s_{i+1}\right)\right).$$

This tridiagonal system is solved by using method of discrete invariant imbedding algorithm which is described in the next section.

3. NUMERICAL EXAMPLES

To validate the efficiency of the method, we applied it to two linear examples and one nonlinear example.

Example 3.1

Consider the singularly perturbed delay differential equation with left layer

$\varepsilon y''(x)+y'(x-\delta)-y(x)=0$; $x\in[0,1]$, with $y(0)=1$ and $y(1)=1$.

The exact solution is given by

y(x)=(1−em2)
em1x+(em1−1)em2xem1−em2y(x)=(1−em2)em1x+(em1−1)em2xem1−em2where m1=(

−1−1+4(ε−δ)−−−−−−−−−−√)/(2(ε−δ))m1=(−1−1+4(ε−δ))/(2(ε−δ)) and m2=(−1+1+4(ε−δ)−−−−−−−−−−√)/(2(ε−δ))m2=(−1+1+4(ε−δ))/(2(ε−δ)).

Numerical results are presented for $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$, respectively.

Example 3.2

Consider the following singularly perturbed variable coefficient and non-homogenous delay differential equation:

εy″(x)+e−0.5xy′(x−δ)−y(x)=0,0≤x≤1εy″(x)+e−0.5xy′(x−δ)−y(x)=0,0≤x≤1with boundary conditions $y(0) = 1, -\delta \le x \le 0, y(1) = 1$.

The exact solution of the problem is not known. Numerical results are presented in Tables 3 and 4 for $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$, respectively.

4 NONLINEAR PROBLEMS

To solve nonlinear singular perturbation problems, we used the method of quasi-linearization.

Example 4.1

Consider the following non-linear singularly perturbed delay differential equation:

εy″(x)+y(x)y′(x−δ)−y(x)=0εy″(x)+y(x)y′(x−δ)−y(x)=0under the interval and boundary conditions

y(x)=1,−δ≤x≤0,y(1)=1y(x)=1,−δ≤x≤0,y(1)=1

The quasilinear form of this example is

εy″(x)+y′(x−δ)−y(x)=0;y(x)=1,−δ≤x≤0,y(1)=1εy″(x)+y′(x−δ)−y(x)=0;y(x)=1,−δ≤x≤0,y(1)=1

The exact solution of the problem is not known. Numerical results for $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$, respectively.

5 DISCUSSION AND CONCLUSIONS

A terminal boundary-value technique has been presented for solving singularly perturbed delay differential equations whose solutions exhibits boundary layer behaviour. The method is iterative on the terminal point x_p and the process is to be repeated for different values of x_p (the terminal point which is not unique), until the solution profile stabilizes in both the inner and outer regions. The present method has been implemented on two linear and one nonlinear problem with left-end boundary layer, by taking $\delta = 0.1\varepsilon$, $\delta = 0.5\varepsilon$ and different values of ε. The numerical results have been tabulated and compared with the exact solutions. Although the solutions are computed at all the points with mesh size h only a few values have been reported. It can be observed from the tables (Tables 1–6) and figures that the present method approximates the exact solution very well. In fact, the method helps us not only to get good results but also to know the behaviour of the solution in the boundary layer/inner region with $h \geq \varepsilon$ where the existing numerical methods fail to give good results. The method is simple, easy and efficient technique for solving singularly perturbed delay differential equations.

Fig. 1 Inner layer solutions for $\varepsilon = 0.001$, $\delta = 0.1\varepsilon$ and different terminal points.

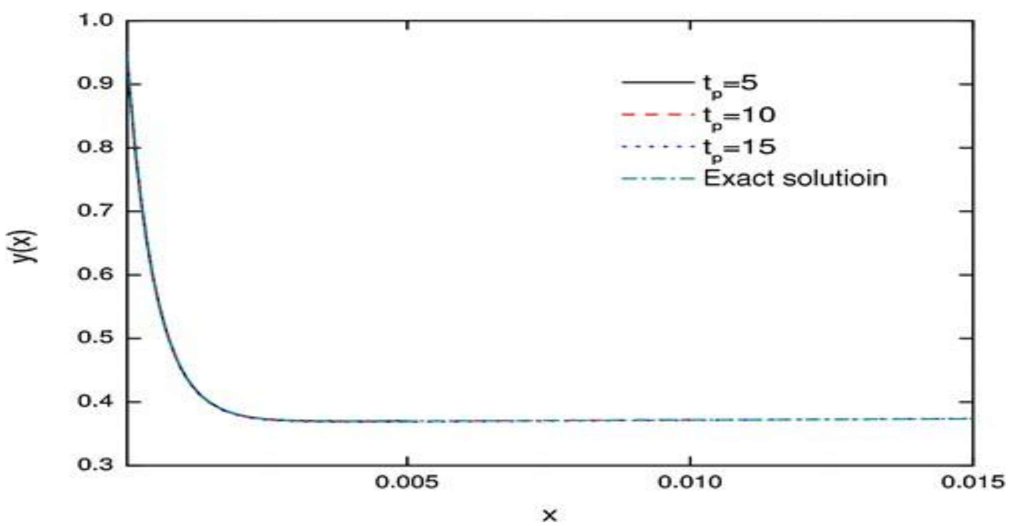


Fig. 2 Inner layer solutions for $\varepsilon = 0.0001$, $\delta = 0.1\varepsilon$ and different terminal points.

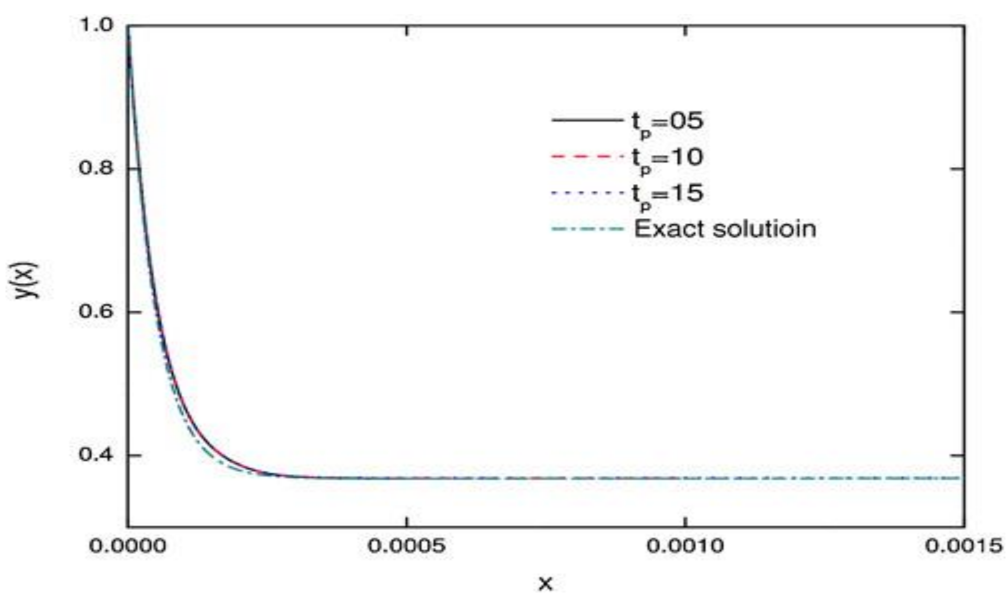


Fig. 3 Inner layer solutions for ε =0.001, δ = 0.1ε and different terminal points.

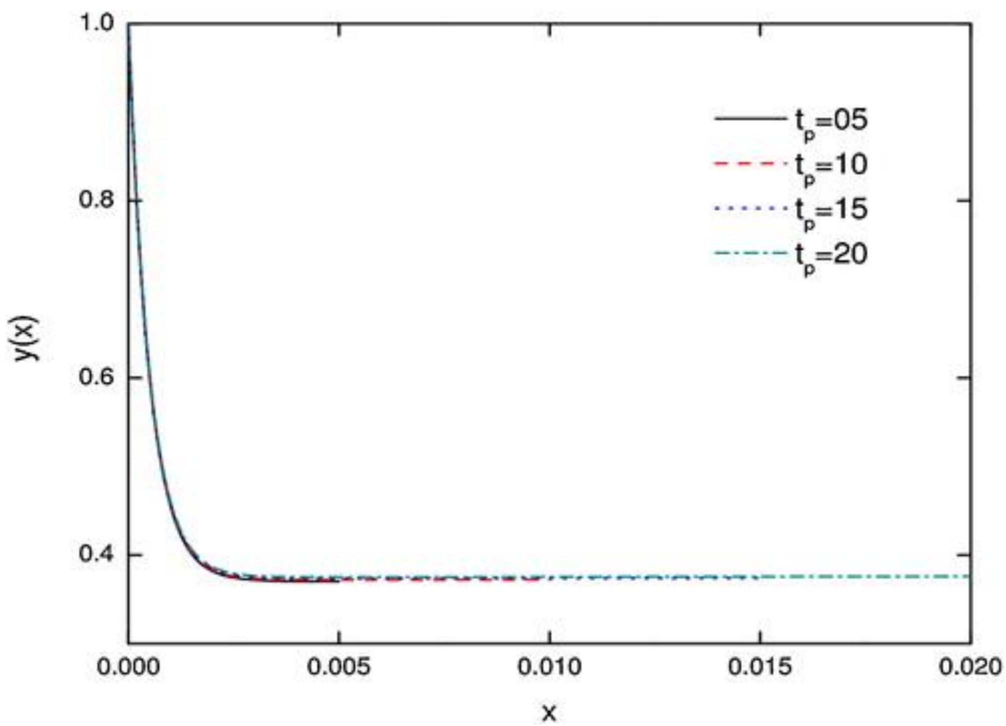
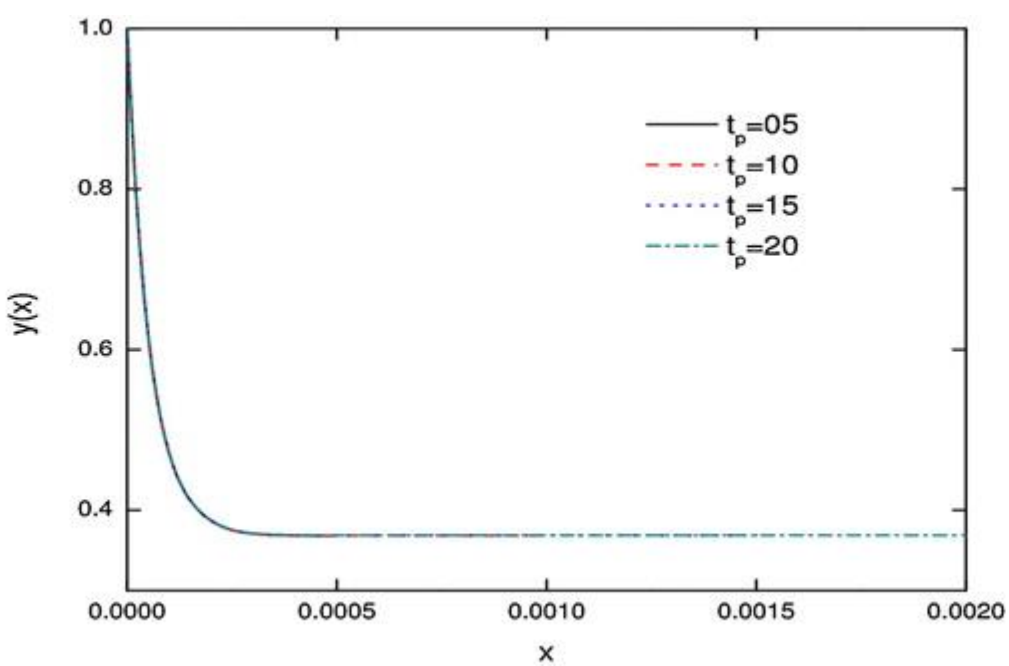


Fig. 4 Inner layer solutions for ε = 0.0001, δ = 0.1ε and different terminal points.



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