UNIFORMITY

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ABSTRACT

In this paper we have studied that a uniform topology induced by a uniform structure on a set is also a topology on the set. Further investigations were carried out to establish the uniform structure which has basically induced a usually topology on the real line R_1.

Keywords : Uniformity, Uniform topology, Uniform structure, product uniformity, topological space, Hausedorff space, metric space

1. INTRODUCTION

The theory of uniform spaces on a non-empty set X has been constructed by A. Weil (1935) [1] in terms of subsets of $X \times X$. J.W. Tukey (1940) [2] later provided as alternative description of a uniform structure using covers of X. After them Bourbaki (1951) [3] defined the uniform space by a certain system of neighbourhoods of diagonal of square $X \times X$. The very general Csaszar (1960) [4] defined the uniform space by a certain ordering of set of all subsets of X. The natural notions of completeness and full-boundness (pre-compactness) are equivalent to the corresponding metric properties for metric space.

2. PRELIMINARIES

In this section we will recall some concepts of uniform spaces.

Definition 2.1. Let X, Y be two sets. If two each element x □□X one can associate one and only one element of Y by some law, then this association is called a ***mapping*** of X into Y.
Definition 2.2. Let X be a non-empty set. A collection of subsets of τ is of subsets of X into be a ***topology*** on X if the following conditions hold: (i) □, X □□□□

(ii) {U□□} □ □□ [ ]:A $\bigcup$ □ □□ U □□□ (iii)

U U_1, 2□ □□ U U_1 $\bigcap$ 2□□.

The order pair (X, □) is called a ***topological space***. Each member U of □□in a Topological space (X, □), is called an ***open set***. The complement of an open set with respect to X is called a closed set.

Definition 2.3. Let (X, □) be a topological spaces and let x □□X. A subset P of X is said to be a ***neighbourhoods*** of X if there exist an open set U□□X such that x □□U □□P.

Definition 2.4. Let (X, □) be a topological space. A be a open set of X, a point x is said to be a ***limit point*** of A if for all open set U containing x

$A \cap$□(U – {x}) □□□.

Here x may or may not be a member of A.

Definition 2.5. Let (X, □) be a topological space. A subfamily □□of □□is said to be a ***base*** of □□if for each x □□X and each U in □□such that x □□U there exist a B in □□such that x □□B □□U, then □□is said to be a ***base*** of □.

A subfamily □□of □□is said to be a ***sub base*** of □□if the family consisting of the finite intersection of sets in □□is a base of □.

Definition 2.6. Let *X* be a set. Let *d* be a real valued function define on the product *X* □□*X* such that

(i) *d*(*x*, *y*) □□0, and = 0 iff *x* = *y*;

(ii) *d*(*x*, *y*) = *d*(*x*, *y*);

(iii) *d*(*x*, *y*) □□*d*(*x*, z) + *d*(z, y), *x*, *y*, *z* □□*X* (triangular inequality).

Such a function *d* is said to be semi metric or ***pseudometric*** and *X* together with *d* is written as (*x*, *d*) is called a ***semi metric*** space. If *d* satisfies only (*i*) and (iii) then *d* is called a ***quasimetric*** and (*X*, *d*) is called ***quasimetric*** space.

If, in addition *d*(*x*, *y*) = 0 iff *x* = *y*, then *d* is called a metric, and *X* together with *d* i.e., is called ***metric space***.

Definition 2.7. ***T_0 space.*** A topological space is said to be ***T_0*-space**, if ***T_0*** separation axiom satisfied i.e., for any distinct point *x*, *y* □*X*, there exist an open set containing one of them but not the other i.e., □*U* □□□such that, x□□*U* and *y* □□*U* or *y* □, *x* □□*U*

Definition 2.8. ***T_1 space*:** A topological space is (*X*, □□) is said to be a ***T_1- space*** if ***T_1*** separation axiom satisfied i.e., for any two distinct points *x*, *y* □*X*, there exist open sets *U* and *V*, where *x* □□*U*, *y* □□*U* or *y* □□*V*, *x* □□*V*.

Definition 2.9. ***T_2-space or Hausedorff space*:** A topological space is (*X*, □□) is said to be a ***T_2-space*** or ***Hausedorff space*** if ***T_2*** separation axiom satisfied i.e., for any two distinct points *x*, *y* □□*X*, there exist open sets *U* and *V*, such that *x*□□*V*, *y* □□*V*, *U* □□*V* = □□.

Definition 2.10. ***Regular space*:** A topological space is (*X*, □□) is said to be ***regular*** if for any closed set *F* and for any *x* □□*F* there exist open sets *U* and *V* such that *x* □□*U*, *F* □□*V* and *U* □□*V* = □.

3. UNIFORMITY AND TOPOLOGY

Now we see that uniformity for a set *E* defines a topology.

Definition 3.1. Let (*E*, {*U*}) be a uniform space. The topology defined by the uniformity {*U*} is the collection of all subsets *T* of *E* such that for each *x* □ *T* there is a *U* □{*U*} with

U[*x*] = {*y* □ *E* : (*x*, *y*) □ *U* } □ *T*.

That the collection ***T*** of all subsets *T* of *E* satisfying the condition in the above definition does indeed define a topology is a simple matter of verification.

The relevant information concerning the open and closed subsets of the topology defined by a uniformity is given in the following:

Theorem 3.2. Let (*E*, {*U*}) be uniform space and Let *T* be the topology on *E* defined by {*U*}.

(i) If {*B*} is a base (or sub base) of the uniformity {*U*}, then the family *U*[*x*], where *U* runs over {*B*} is a base (or sub base) of the neighbourhood filter of *x*. Hence each *x* □ *E* has a base of neighbourhood filters, each member of which is symmetric (i.e., when *U* in {*U*} is symmetric).

(ii) If A^o is the ***T***-interior of a subset *A* □ *E*, then A^o = {*x* □ *E* : for some *U* □ **U**, *U*[*x*] □ *A*}.

(iii) If $\bar{A}$ is the ***T***-closure of *A* □ *E*, then *A* U $\bar{A}$ ∩ U *A*□ { [ ]: □ }.

Proof. (i) If ***B*** be a base (or sub base) for the uniformity **U**, then for each *x* □ *E*, the family {*U*[*x*]}, where *U* runs over **U**, forms a base (or sub base) for the neighbourhood filter of *x*. Consequently the symmetric neighbourhoods *U*[*x*] form a base of the neighbourhood filter of *x* and *U*[*x*] ∩ U^{-1}[*x*] is a symmetric neighbourhood of *x*.

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(ii) Put B = {x □ E: for some U □ **U**, $U[x]$ □ A}. Then for each x □ B, there exists U □ **U** such that $U[x]$ □ B. Also there exists V □ **U** such that $V^2 = V$ o V □ U. To show that $V[x]$ □ B, let y

□ $V[x]$. Then $V[y]$ □ $V^2][x]$ □ $U[x]$ □ A. Hence y □ B. But since $V[x]$ is ***T*** open. It is now clear that each T-open subset C of A is contained in B since C contains a subset of the type $U[x]$. Hence $B = A^o$.

(iii) $x\bar{A}$□ if and only if, for each U □ **U**, $U[x] \cap \bar{A}$ □ □ if and only if x □ U^{-1}. Since U □ **U** contains a symmetric member, it follows that $x A$□ if and only if x □ $U[A]$ for each U □ **U**. Therefore, $A\ U \bigcap A\ U$□ { [ ]: □**U**}. **4. PRODUCT TOPOLOGY**

Definition 4.1. Just as the Cartesian product of topological spaces can be given a topology so the Cartesian product of uniform spaces can be given uniformity. Specially, let $\{E_j\}$ be a family of uniform spaces. We consider the Cartesian product E = □ E_j. We identify $E \times E$ with the Cartesian product □ $(E_j \times E_j)$. Now take the Cartesian product □ U_j, where U_j □ $E_j \times E_j$ for each index j, as a subset of $E \times E$. Now consider the family of subsets of $E \times E$ consisting of the restricted product set □ U_j, where U_j is an entourage of E_j for each index j. This family constitutes a base for a uniformity, is called the ***product uniformity,*** and E, with this uniformity, is called the ***uniform product.***

5. UNIFORMITY AND SEPARATION AXIOMS

We have seen that how a uniformity on E^2 induces a topology on E and how the product topology on E^2 is define by the topology on E. It is natural to expect some relation between the uniformities and the separation axioms define for the associated topologies.

Theorem 5.1. Let (E, **U**) be a topological space, where the topology induced by a uniformity **U**. Then (E, **U**) is a T_3-space. Hence if (E, **U**) is a T_1-space, then (E, **U**) is regular.

Proof. Let x □ E. For each neighbourhood $U[x]$ of x, there exist a V of U such that V o V □ **U**. Then $\bar{V}\ x[\ ]$□$\bigcap\{W \circ V\ x\ W[\ ]$: □**U**} is a closed neighbourhood of x and $V\ x\ \bar{U}\ x[\ ]$□ [ ]. By characterization of regular spaces it follows that (E, **U**) is a T_3-space.

Now to show remaining proof let us go to the following theorem.

Theorem 5.2. Let (E, **U**) be a uniform space and let ***u*** be the topology define by **U** on E. The following are equivalent

(i) (E, **U**) is a T_1 space;

(ii) (E, **U**) is a Hausdorff space;

(iii) ∩ {U : U □ **U**} = □, the diagonal set. (iv) (E, **U**) is regular.

Proof. We first show (iv) □ (ii)

Let x, y □ E, x □ y. By hypothesis x has an open neighbourhood U which does not contain y. Then by regularity there exist a open neighborhood V of x such that $x\ \bar{V}\ V\ U$□ □ □ . Since y does not contain in U, it follows that $y\ \bar{V}$□ .. This shows that $y\ E\ \bar{V}$□ \ .. Since $V\ (E \bigcap \bar{V} \backslash\ )$ □□ and V, $E\backslash V$ are open neighborhood of x and y respectively, this proves that E is a Hausedorff space.

So we have (E, **U**) is Hausedorff space.

Now by separation axiom we have directly that a Hausedorff space is T_1-space. That has been proved.

So (i) □ (iv) by above theorem. Hence (i) □ (ii) □ (iv).

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