Lindley Approximation of ScaleParameters of Weibull Model with Precautionary Loss Function

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Abstract

This paper describes the unknown parameter the Weibull distribution based Approximating Bayesian model for the Type-II censored data. The scale parameter of the Weibull distribution is measured with a natural conjugate gamma prior for known shape parameter. Furthermore, the scale parameter of the gamma prior is assumed to be two different known hyperparameters. Under these assumptions, the Weibull parameter are estimated on the precautionary loss function(PLF), which is asymmetric and can be easily extended to other loss functions in failure situation. The result from Approximation by Bayesian method is used to compare with Bayes and maximum likelihood estimate (MLE) methods. The simulation shows the results from Bayes is the best in comparison with estimators obtained by usingApproximate Bayesian method, and then MLE in terms of root mean square error (RMSE).

Keywords: Lindley Approximation, Bayes Estimator, Weibull Model, Precautionary Loss, Type II Censoring.

Introduction

The Weibull distribution is mostly used in reliability analysis and life data analysis because of its ability to adapt to different situations. Depending upon the parameter values, this distribution is used to model the variety of behavior's for a particular function. The probability density function usually describes the distribution function. The parameters in the distribution control the shape and scale of the probability density function. Several methods are used to measure the reliability of the data. But the Weibull distribution method is one of the best methods to analyses the life data.The distribution is named after the Swedish scientist Weibull who proposed if for the first time in 1939 in connection with his studies on strength of material. Weibull (1951) showed that the distribution is also useful in describing the wear out of fatigue failures. Estimation and properties of the Weibull distribution is studied by many authors [see Kao (1959)].

The probability density function reliability and hazard rate functions of Weibull distribution aregiven respectively as

$$f(x) = \mu\nu x^{(\mu-1)} \exp(-\mu\, x^{\mu}) \quad ; \quad x, \nu, \mu > 0 \qquad (1.1)$$

$$R(t) = exp(-\nu t^{\mu}) \quad ; \quad t > 0 \qquad (1.2)$$

$$H(t) = \mu\nu t^{(\mu-1)} \quad ; \quad t > 0 \qquad (1.3)$$

Where $'\nu'$ is the scale and 'μ' is shape parameters.

The most widely used loss function in estimation problems is Squared error loss function given as

$$L(\widehat{\Delta}, \Delta) = (\widehat{\Delta} - \Delta)^2 \qquad (1.4)$$

known as squared error loss function (SQELF). This loss function is symmetrical because it associates the equal importance to the losses due to overestimation and under estimation with equal magnitudes however in some estimation problems such an assumption may be inappropriate. Overestimation may be more serious than underestimation or Vice-versa Ferguson(1967). Canfield (1970), Basu and Ebrahimi(1991). Zellner (1986) Soliman (2000) derived and discussed the properties of varian's (1975) asymmetric loss function for a number of distributions.

Norstrom (1996) introduced an alternative asymmetric precautionary loss function and also presented a general class of precautionary loss function with quadratic loss function as a special case. These loss function approach infinitely near the origin to prevent underestimation and thus giving a conservative estimator, especially when, low failure rates are being estimated. These estimators are very useful and simple.

The asymmetric precautionary loss function is given as

$$L(\hat{\nu}, \nu) = \frac{(\hat{\nu} - \nu)^2}{\hat{\nu}} \qquad (1.5)$$

where $\hat{\nu}$ is an estimate of ν .

The posterior expectation of the precautionary loss function in equation (1.5)is

$$E_\pi\left[\frac{(\hat{\nu} - \nu)^2}{\hat{\nu}}\right] = E_\pi(\hat{\nu}) - 2E_\pi(\nu) + E_\pi\left(\frac{\nu^2}{\hat{\nu}}\right), \quad (1.6)$$

The Bayes estimator $\hat{\nu}_{BPL}$ of ν under precautionary loss function is the value of $\hat{\nu}$ which minimizes equation(1.6) is

$$\hat{\nu}_{BPL} = [E_\pi(\nu^2)]^{\frac{1}{2}}, \quad (1.7)$$

Provided that $E_\pi(\nu^2)$ exists and is finite.

In a Bayesian setup, the unknown parameter is viewed as random variable. The uncertainty about the true value of parameter is expressed by prior distribution. The parametric inference is made using the posterior distribution which is obtained by incorporating the observed data into the prior distribution using the Bayes theorem, the first theorem of inference. Hence, we update the prior distribution in the light of observed data. Thus, the uncertainty about the parameter prior to the experiment is represented by the prior distribution and the same after the experiment is represented by the posterior distribution.

The Estimators

Let $x_1, x_2, \dots \dots \dots x_n$ be the life times of 'n' items that are put on test for their lives, follow a weibull distribution with density given in equation (1.1). The failure times are recorded as they occur until a fixed number 'r' of times failed. Let $= (x_{(1)}, x_{(2)}, \dots \dots \dots \dots \dots, x_{(n)})$, where $x_{(i)}$ is the life time of the ith item. Since remaining (n-r) items yet not failed thus have life times greater than $x_{(r)}$.

The likelihood function can be written as

$L(x|\nu,\mu) = \frac{n!}{(n-r)!}(\mu\nu)^r \prod_{i=1}^{r} x_i^{(\mu-1)} \exp(-\delta\nu)$,(2.1)

Where $\delta = \sum_{i=1}^{r} x_i^{\mu} + (n-r)x_r^{\mu}$

The logarithm of the likelihood function is

$\log L(x|\nu,\mu) \propto r \log \mu + r \log \nu + (\mu-1)\sum_{i=1}^{r} \log x_i - \delta\nu$,(2.2)

assuming that 'μ' is known, the maximum likelihood estimator $\hat{\nu}_{ML}$of ν can be obtain by using equation (2.2) as

$\hat{\nu}_{ML} = r/\delta$ (2.3)

In case if both the parameters μ and ν are unknown their MLE's $\hat{\mu}_{ML}$and $\hat{\nu}_{ML}$ can be obtained by solving the following equation

$\frac{\delta}{\delta\nu}\log L = \frac{r}{\nu} - \delta = 0$, (2.4a)

$\frac{\delta \log L}{\delta\mu} = \frac{r}{\mu} + \sum_{i=1}^{r} \log x_i - \nu\delta_1 = 0$,(2.4b)

where

$\delta_1 = \sum_{i=1}^{r} x_i^{\mu} \log x_i + (n-r)x_r^{\mu} \log x_r$, eliminating ν between the two equations of (2.4) and simplifying we get

$\hat{\mu}_{ML} = \frac{r}{\delta^*}$ (2.5)

Where $\delta^* = \left[\frac{r\delta_1}{\delta} - \sum_{i=1}^{r} log x_i\right]$

Equation (2.5) may be solved for Newton- Raphson or any suitable iterative Method and this value is substituted in equation (2.4b) by replacing with μ get $\hat{\mu}$ as

$\hat{\nu}_{ML} = \frac{\frac{r}{\hat{\mu}_{ML}} + \sum_{i=1}^{r} log x_i}{\sum_{i=1}^{r} x_i^{\hat{\mu}_{ML}} \; log x_i + (n-r)x_r^{\hat{\mu}_{ML}} \; log x_r}$,(2.6)

3. Bayes Estimator of Scale Parameter ν when shape Parameter μ is known.

If μ is known assume gamma prior $\rho(c,d)$ as cojugate prior for ν as

$\phi(\nu|\underline{x}) = \frac{d^c}{\Gamma c}(\nu)^{(c+1)} \exp(-d\nu)$; $(c,d) > 0, \nu > 0$,(3.1)

The posterior distribution of ν using equation (2.1) and (3.1) we get

$\psi(\theta|\underline{x}) = \frac{(\delta+d)^{r+c}}{\Gamma(r+c)}(\nu)^{(r+c-1)} \exp(-\nu(\delta+d))$, (3.2)

Under General Precautionary Loss Function, the Bayes estimator $\hat{\nu}_{BPL}$ of ν using (1.5) and (3.2) given by

$\hat{\nu}_{BPL} = \left[\frac{(r+a)(r+a+1)}{(\delta+d)}\right]^{\frac{1}{2}}$(3.3)

4. The Bayes estimators with ν and μ unknown.

The joint prior density of ν and μ is given by

$\phi^*(\nu|p) = \phi_1(\nu|\mu).\phi_2(\mu)$

$\phi^*(\nu|\mu) = \frac{1}{\lambda\Gamma\xi} p^{-\xi} \nu^{(\xi-1)}.\exp\left[-\left(\frac{\nu}{\mu} + \frac{\mu}{\lambda}\right)\right]$; $(\nu,\mu,\lambda,\xi) > 0$, (4.1)

where

$\phi_1(\nu|\mu) = \frac{1}{\Gamma\xi}\nu^{-\xi}\nu^{(\xi-1)}.\exp\left[-\frac{\nu}{\mu}\right]$;(4.2)

And

$\phi_2(\mu) = \frac{1}{\lambda}\exp\left(-\frac{\mu}{\lambda}\right)$; (4.3)

The joint posterior density of ν and μ is

$$\psi^*(\nu,\mu|\underline{x}) = \frac{\frac{1}{\lambda\Gamma\xi}p^{-\lambda}\nu^{(\xi+1)}\exp\left[-\left\{\frac{\nu}{\mu}+\frac{\mu}{\lambda}\right\}\right](\nu\mu)^r\prod_{i=1}^{r}x_i^{(\mu-1)}e^{-\mu\nu}}{\iint\frac{1}{\lambda\Gamma\xi}\mu^{(r-\xi)}\nu^{(r+\xi+1)}\prod_{i=1}^{r}x_i^{(\mu-1)}.\exp\left[-\left\{\frac{\nu}{\mu}+\frac{\mu}{\lambda}+\mu\nu\right\}\right]d\nu d\mu}; \quad (4.4)$$

Approximate Bayes Estimators

The Bayes estimators of a function $\rho = \rho(\nu,\mu)$ of the unknown parameter ν *and* μ under squared error loss is the posterior mean

$\hat{\varrho}_{ABS} = E(\mu|\underline{x}) = \frac{\iint\phi(\nu\mu)\phi^*(\nu,\mu|\underline{x})d\nu d\mu}{\iint\phi^*(\nu,\mu|\underline{x}).d\nu d\mu}$;(4.5)

By using Lindley approximation method we evaluate equation(4.5)

$E(\varrho(\nu,\mu)|\underline{x}) = \frac{\int\phi(\nu).e^{(l(\nu)+\varrho(\nu))}d\nu}{\int e^{(l(\nu)+\varrho(\nu))}d\nu}$; (4.6)

Where $l(\nu) = \log\phi(\nu)$, and $\phi(\nu)$ is an arbitrary function of ν and $l(\nu)$ is the logarithm likelihood function

The Lindley approximation(Lindley (1980)) for two parameter is

$E(\hat{\varrho}(\nu,\mu)|\underline{x}) = \varrho(\nu,\mu) + \frac{A}{2} + \rho_1 A_{12} + \rho_2 A_{21} + \frac{1}{2}[l_{30}B_{12} + l_{21}C_{12} + l_{12}C_{21} + l_{03}B_{21}]$,(4.7)

where

$$A = \sum_1^2\sum_1^2 \varrho_{ij}\sigma_{ij} \; ; \qquad l_{\eta\epsilon} = \left(\delta^{(\eta+\epsilon)}l\middle|\delta\nu_1^{\eta}\delta\nu_2^{\epsilon}\right);$$

where$(\eta+\epsilon) = 3$ for $i,j = 12$ $\rho_i = (\delta\rho|\delta\nu_i)$;

$$\varrho_i = \frac{\delta\varrho}{\delta\nu_i}\; ; \;, \quad \varrho_{ij} = \frac{\delta^2\varrho}{\delta\nu_i\delta\nu_j}\; ;\forall i \neq j\; ;$$

$$A_{ij} = \varrho_i\sigma_{ij} + \varrho_j\sigma_{ji}\; ; \quad B_{ij} = (\varrho_i\sigma_{ii} + \varrho_j\sigma_{ij})\sigma_{ii}\; ;$$

$$C_{ij} = 3\varrho_i\sigma_{ii}\sigma_{ij} + \varrho_j(\sigma_{ii}\sigma_{jj} + 2\sigma_{ij}^2);$$

Where σ_{ij} is the (i,j)th element of the inverse of matrix $\{-l_{jj}\}; i,j = 1,2$ s.t. $l_{ij} = \frac{\delta^2 l}{\delta\nu_i\delta\nu_j}$.

All the above function are evaluated at MLE of (ν_1,ν_2).In our case $(\nu_1\nu_2) = (\nu,\mu); So\ \phi(\nu) = \phi(\nu,\mu)$

To apply Lindley approximation (4.5), we first obtain σ_{ij} , elements of the inverse of $\{-l_{jj}\}; i,j = 1,2$,which can be shown to be

$\sigma_{11} = \frac{M}{D},\ \sigma_{12} = \sigma_{21} = \frac{\delta_1}{D},\sigma_{22} = \frac{r}{D\,\theta^2}$,

Where$M = \left(\frac{r}{\mu^2} + \nu\delta_2\right)$;$D = \left[\frac{r}{\nu^2}\left(\frac{r}{\mu^2} + \nu^2\delta_2\right)\right]$;

$\delta_2 = \sum_{i=1}^{r} x_i^{\mu}(log x_i)^2 + (n-r)x_r^{\mu}(log x_r)^2$;

To evaluate ρ_i , take the joint prior $\phi^*(\nu|\mu)$

$\phi^*(\nu|\mu) = \frac{1}{\lambda\Gamma\xi}\mu^{-\xi}\nu^{(\xi-1)}.\exp\left[\left\{-\frac{\nu}{\mu}+\frac{\mu}{\lambda}\right\}\right]$; $(\nu,\mu,\lambda,\xi) > 0$, (4.9)

$\Rightarrow \rho = \log[\phi^*(\nu|\mu)] = constant - \xi log\mu - (\xi-1)log\nu - \frac{\nu}{\mu} - \frac{\mu}{\lambda}$

Therefore

$$\rho_1 = \frac{\partial\rho}{\partial\nu} = \frac{(\xi-1)\nu}{\nu} - \frac{1}{\nu};$$

and

$$\rho_2 = \frac{\nu}{\mu^2} - \frac{1}{\lambda} - \frac{\xi}{\mu};$$

Further more

$l_{21} = 0\,; l_{12} = -\delta_2\,; l_{03} = \frac{2r}{p^3} - \nu\delta_3; \, and \;\; l_{30} = \frac{2r}{\nu^3};$

Where $\delta_3 = \sum_{i=1}^{r} x_i^{\nu}\,(logx_i)^3 + (n-r)x_r^{\nu}(logx_r)^3$

By substituting above values in equation(4.7), yields the Bayes estimator under SQELF using Lindley approximation denoted by $\hat{\varrho}_{ABS}$

$\hat{\varrho}_{ABSQ} = E\big(\varrho(\nu,\mu)\big) = \varrho(\nu,\mu) + \mathrm{U} + \varrho_1 U_1 + \varrho_2 U_2;$ (4.10)

Where U $=\frac{1}{2}[\varrho_{11}\sigma_{11} + \varrho_{21}\sigma_{21} + \varrho_{12}\sigma_{12} + \varrho_{22}\sigma_{22}];$ (4.10a)

$U_1 = \frac{1}{\nu^2 D^2}[\frac{M\nu D}{\mu}(\mu(\xi-1)-1) + \frac{\nu^2\delta_1 D}{\lambda\mu^2}\{\lambda\nu - \mu^2 - \lambda\xi\mu\}$

$+\frac{rM^2}{\nu} - \frac{rM\delta_1}{2} - \nu^2\delta_1^2\delta_2 + \frac{r^2}{\nu^3}\delta_1 - \frac{\nu r\delta_1\,\delta_3}{2}];$ (4.10b)

$U_2 = \frac{1}{\nu^2 D^2}[\frac{\nu\;\;\delta_1 D}{\mu}(\mu(\xi-1)-\nu) + \frac{rD}{\lambda\mu^2}\{\lambda\nu - \mu^2 - \lambda\xi\mu\}$

$+\frac{rM\delta_1}{\nu} - \frac{3\delta_1 r\delta_2}{2} + \frac{r^2}{\nu^2\mu^3} - \frac{r^2\delta_3}{2\nu}];$ (4.10c)

All the function of right-hand side of the equation (4.10) are to be evaluated for $\hat{\nu}_{ML}$ and $\hat{\mu}_{ML}$.

Approximate Bayes Estimators under Precautionary loss function

The Approximate Bayes estimator of a function $\varrho = \varrho(\nu,\mu)$ of unknown parameters ν *and* μ under PLF in equation (1.7) is given by

$\hat{\varrho}_{ABP} = [E_h(\varrho^2)]^{\frac{1}{2}}$ (4.11)

Where

$E_{h^*}(\varrho^2|\bar{x}) = \frac{\iint \varrho^2\,\psi^*(\nu,\mu)d\theta\,dp}{\iint\;\psi^*(\nu,\mu)d\theta\,dp}$;(4.12)

Special Cases

(1) Let$\varrho(\nu,\mu) = \frac{1}{\nu}$;

The approximate Bayes Estimator of under Precautionary loss function is

$\hat{\nu}_{ABPL} = \left[\nu^2 + \frac{M}{D} + 2\nu U_1\right]^{\frac{1}{2}}$; at $(\hat{\nu}_{ML}, \hat{\mu}_{ML})$, (4.13)**Numerical Calculations and Comparison.**

The numerical calculations are done by using R Language programming and results are presented in form of tables.

1. The values of $(\nu,\mu)$and are generated from the equations (4.3 - 4.4) for given c=2, and d=3, which comes out to be ν=0.238 and μ=0.227. For these values of and μ the Weibull random variates are generated.
2. Taking the different sizes of samples n=25 (25) 100 with failure censoring, MLE's, the Approximate Bayes estimators,and their respective MSE's (in parenthesis) by repeating the steps 500 times, are presented in the tables from (1), for parameters of prior distribution c =2 , and d =3.
3. Table (1) presents the MLE of parameter of (for known μ) and approximate Bayes estimators under PLF (for ν and μ both unknown). The MSE's in all above cases are presented in parenthesis. The estimators have minimum MSE's for small sample sizes, as the sample sizes increase, the MSE's increased. Among all the four estimators $\hat{\nu}_{ABPL}$under PLF has the lowest MSE.

Table(1)

Mean and MSE's of ν

$(\lambda = 2, \xi = 3, = .238, \mu = .227, a = 20)$

n	r	$\hat{\nu}_{ML}$	$\hat{\nu}_{BPL}$	$\hat{\nu}_{ABPL}$
25	20	0.032417	0.0227771	0.0278503
		(9.2278x10^{-5})	**((7.9384x10^{-5})**	**(7.8801x10^{-5})**
50	30	0.0470689	0.0375839	0.0279253
		(5.8866x10^{-5})	**(4.5669x10^{-5})**	**(4.3665x10^{-5})**
75	50	0.2835593	0.3773257	0.3829046
		(8.4077x10^{-4})	**(5.9679x10^{-4})**	**(7.2492x10^{-4})**
100	75	0.3801292	0.3755629	0.3801086
		(7.5996x10^{-4})	**(5.5588x10^{-4})**	**(6.5915x10^{-4})**

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